Inverse Function Theorem

1. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x^2 - y^2 \\ xy \end{bmatrix}$$

   (i) Show that F is a local diffeomorphism at any point other than the origin.

   (ii) Suppose $\mathbf{c} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{b} = F(\mathbf{c})$, F is a diffeomorphism on the ball $B_r(\mathbf{c})$ and G is the inverse of $F|_{B_r(\mathbf{c})}$. Find $J_{\mathbf{b}}(G)$

2. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} e^x \cos(y) \\ e^x \sin(y) \end{bmatrix}$$

   (i) Show that F is a local diffeomorphism at any point.

(ii) Show that F is not a (global) diffeomorphism.

3. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x+z \\ y+z^2 \\ x^2+yz \end{bmatrix}$$

(i) Show that F is a diffeomorphism in a neighborhood B of $\mathbf{c} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$.

(ii) Suppose $\mathbf{b} = F(\mathbf{c})$ and G is the inverse of $F|_{B_r(\mathbf{c})}$. Find $J_{\mathbf{b}}(G)$