Implicit Function Theorem

1. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}$

$$F(\begin{bmatrix} x \\ y \end{bmatrix}) = x^2 + y^2 - 25$$

   (i) Show that there is an open interval B centered at the point $b = 4$, an open interval W containing $a = -3$, and a continuously differentiable function $\phi : B \to W$ with $b = \phi(a)$ such that, for all $y \in B$, $x \in W$ and $F(x,y) = 0$ if and only if $x = \phi(y)$.

   (ii) Find the Jacobian $J_{\mathbf{b}}(\phi)$.

   (iii) Does the claim in part (i) hold if we assume $a = 0$ and $b = 5$?

2. Consider the function $F : \mathbb{R}^3 \to \mathbb{R}$

$$F(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = x^3 + 12xz + y^2$$

(i) Show that there is an open disc B centered at the point $\mathbf{b} = (5, 2)$, an open interval W containing $a = -1$, and a continuously differentiable function $\phi : B \to W$ with $a = \phi(\mathbf{b})$ such that, for all $(y,z) \in B$, $x \in W$ and $F(x,y,z) = 0$ if and only if $x = \phi(y,z)$.

(ii) Find the Jacobian $J_{\mathbf{b}}(\phi)$.