Problem Set 1

1. Define addition and scalar multiplication on $\mathbb{R}^2$ as follows:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{a_1} + \sqrt[3]{b_1})^3 \\ (\sqrt[3]{a_2} + \sqrt[3]{b_2})^3 \\ (\sqrt[3]{a_3} + \sqrt[3]{b_3})^3 \end{bmatrix}, \qquad \lambda \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \lambda^3 a_1 \\ \lambda^3 a_2 \\ \lambda^3 a_3 \end{bmatrix}.$$

Show that $\mathbb{R}^2$ together with these operations defines a vector space.

2. For a vector x in a vector space V and a scalar $\lambda \in \mathbb{R}$, let $\lambda v = 0$. Show that either $\lambda = 0$ or $v = 0$.

3. Are the following subsets of $C(\mathbb{R},\mathbb{R})$ subspaces? Justify your answer in each case.

   (i) $W_1 := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } f(1) = 0\}$.

   (ii) $W_2 := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous and } f(0) = 1\}$.

4. The transpose of a matrix $A = [a_{i,j}]$ in $M_{m\times n}(\mathbb{R})$ is the matrix $A^t = \left[a^t_{i,j}\right]$ in $M_{n\times m}(\mathbb{R})$ such that

$$a^t_{i,j} = a_{j,i}.$$

A matrix $A \in M_{n\times n}(\mathbb{R})$ is *anti-symmetric* if $A = -A^t$.

(i) Write down the general form of an anti-symmetric matrix in $M_{3\times 3}(\mathbb{R})$.

(ii) Show that the set of all anti-symmetric matrices is a subspace of $M_{n\times n}$.

5. A *field* is a set F together with the maps $+ : F \times F \to F$ and $\cdot : F \times F \to F$ which satisfy the following properties:

(F1) $a + b = b + a$ for any $a, b \in F$;

(F2) $(a + b) + c = a + (b + c)$ for any $a, b, c \in F$;

(F3) There is $0 \in F$ such that for $a \in F$, we have $a + 0 = a$;

(F4) For any $a \in F$, there is $a' \in F$ such that $a + a' = 0$;

(F5) $a \cdot b = b \cdot a$ for any $a, b \in F$;

(F6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c \in F$;

(F7) There is $1 \in F$ such that for $a \in F$, we have $a \cdot 1 = a$;

(F8) For any non-zero $a \in F$, there is $a'' \in F$ such that $a \cdot a'' = 1$;

(F9) $(a + b) \cdot c = a \cdot c + b \cdot c$ for any $a, b, c \in F$.

(i) Let S be a set with two elements. Define operations $+$ and $\cdot$ such that the resulting space is a field. We denote this field by $\mathbb{Z}/2$. (You do not need to write down the verification of the above properties.)

(ii) We may define the notion of a vector space V over a field F in the same way as before, except that scalar multiplcation is defined as the multiplication of an element of F and a vector V. In particular, the vector addition and scalar multiplication need to satisfy the eight properties that we listed in the class. Let $M_{m\times n}(F)$ be the space of m by n matrices such that each entry is an element of F. Define a vector space structure on $M_{m\times n}(F)$ generalizing the definition of addition and scalar multiplication for $M_{m\times n}(\mathbb{R})$.

(iii) How many elements does the vector space $M_{m\times n}(\mathbb{Z}/2)$ have?