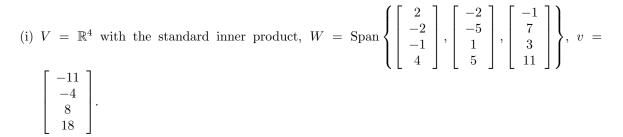
## Problem Set 10

- 1. In each part, apply the Gram-Schmidt process to obtain an orthonormal basis  $S_1$  from the basis  $S_0$  for the vector space V. (You don't need to show that  $S_0$  is a basis.)
  - (i)  $V = \mathbb{R}^3$  with the standard inner product,  $S_0 = \left\{ \begin{bmatrix} 2\\ -2\\ 1 \end{bmatrix}, \begin{bmatrix} 0\\ 1\\ 1\\ 0 \end{bmatrix} \right\}, \begin{bmatrix} 1\\ 1\\ 0\\ 0 \end{bmatrix} \right\}.$

(ii) 
$$V = M_{2 \times 2}(\mathbb{R})$$
 with the standard inner product,  $S_0 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \right\}$ .

(iii)  $V = \mathcal{P}_3$ , polynomials of degree at most 3, with the inner product  $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$ and  $\mathcal{S}_0 = \{1, x, x^2, x^3\}$ . 2. In each part, firstly find an orthonormal basis for the subspace W of V. Then use the basis to write the vector v as a sum w + z where  $w \in W$  and  $z \in W^{\perp}$ .



(ii)  $V = \mathcal{P}_3$ , polynomials of degree at most 3, with the inner product  $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$ ,  $W = \{1, x, x^2\}$  and  $v = x^3 + 2x^2 - 2$ . 3. Suppose  $u_1$  and  $u_2$  are vectors of an inner product space V. Define  $T(v) = \langle v, u_1 \rangle u_2$ . Show that T is a linear map and find its adjoint.

4. Suppose  $A = \{v_1, \ldots, v_k\}$  is a set of vectors in the vector space V with the inner product  $\langle , \rangle$ . Show that if  $\langle w, v_i \rangle = 0$ , then w belongs to the orthogonal complement of Span(A). (We use this fact several times in the class, and here you verify it.)

5. Suppose  $V_1$  and  $V_2$  are vector spaces with inner product. Suppose  $T: V_1 \to V_2$  is a linear transformation and  $T^*: V_2 \to V_1$  is its adjoint. The goal of this problem is to show that

$$\operatorname{kernel}(T^*) = \operatorname{image}(T)^{\perp}, \qquad \operatorname{image}(T^*) = \operatorname{kernel}(T)^{\perp} \tag{1}$$

(i) Using the defining relation

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle$$
 for all  $u \in V_1$  and  $v \in V_2$ ,

of  $T^*$  show that if u is in the kernel of T, then it is orthogonal to any vector in the image of  $T^*$ . This shows that  $\operatorname{kernel}(T) \subseteq \operatorname{image}(T^*)^{\perp}$ .

(ii) Show that if u is orthogonal to any vector in the image of  $T^*$  (a vector of the form  $T^*(v)$ ), then u is in the kernel of T. Using this and the last part, show that  $\text{kernel}(T) = \text{image}(T^*)^{\perp}$ .

(iii) Derive the identities in (1) from the result of the last part using the relations  $(W^{\perp})^{\perp} = W$  and  $T^{**} = T$ .

6. Suppose V is a vector space over complex numbers  $\mathbb{C}$  with an inner product  $\langle , \rangle : V \times V \to \mathbb{C}$ . Since  $\mathbb{R} \subset \mathbb{C}$ , the vector space V can be also regarded as a vector space over real numbers  $\mathbb{R}$ . For any two vector  $u, v \in V$ , let  $[u,v] \in \mathbb{R}$  be the real part of  $\langle u,v \rangle \in \mathbb{C}$ . Show that if V is regarded as a vector space over  $\mathbb{R}$ , then [,] defines an inner product on V. Show also that  $[v,\sqrt{-1}v] = 0$ .