Problem Set 10

1. In each part, apply the Gram-Schmidt process to obtain an orthonormal basis $\mathcal{S}_1$ from the basis $\mathcal{S}_0$ for the vector space V. (You don't need to show that $\mathcal{S}_0$ is a basis.)

   (i) $V = \mathbb{R}^3$ with the standard inner product, $\mathcal{S}_0 = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   (ii) $V = M_{2\times 2}(\mathbb{R})$ with the standard inner product, $\mathcal{S}_0 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \right\}$.

   (iii) $V = \mathcal{P}_3$, polynomials of degree at most 3, with the inner product $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$ and $\mathcal{S}_0 = \{1, x, x^2, x^3\}$.

2. In each part, firstly find an orthonormal basis for the subspace W of V. Then use the basis to write the vector v as a sum $w + z$ where $w \in W$ and $z \in W^{\perp}$.

(i) $V = \mathbb{R}^4$ with the standard inner product, $W = \text{Span}\left\{ \begin{bmatrix} 2 \\ -2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \\ 3 \\ 11 \end{bmatrix} \right\}$, $v = \begin{bmatrix} -11 \\ -4 \\ 8 \\ 18 \end{bmatrix}$.

(ii) $V = \mathcal{P}_3$, polynomials of degree at most 3, with the inner product $\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$, $W = \{1, x, x^2\}$ and $v = x^3 + 2x^2 - 2$.

3. Suppose u_1 and u_2 are vectors of an inner product space V. Define $T(v) = \langle v, u_1 \rangle u_2$. Show that T is a linear map and find its adjoint.

4. Suppose $A = \{v_1, \ldots, v_k\}$ is a set of vectors in the vector space V with the inner product $\langle\, , \rangle$. Show that if $\langle w, v_i \rangle = 0$, then w belongs to the orthogonal complement of Span(A). (We use this fact several times in the class, and here you verify it.)

5. Suppose V_1 and V_2 are vector spaces with inner product. Suppose $T : V_1 \to V_2$ is a linear transformation and $T^* : V_2 \to V_1$ is its adjoint. The goal of this problem is to show that

$$\text{kernel}(T^*) = \text{image}(T)^\perp, \qquad \text{image}(T^*) = \text{kernel}(T)^\perp \tag{1}$$

(i) Using the defining relation

$$\langle T(u), v\rangle = \langle u, T^*(v)\rangle \qquad \text{for all } u \in V_1 \text{ and } v \in V_2,$$

of T^* show that if u is in the kernel of T, then it is orthogonal to any vector in the image of T^*. This shows that $\text{kernel}(T) \subseteq \text{image}(T^*)^\perp$.

(ii) Show that if u is orthogonal to any vector in the image of T^* (a vector of the form $T^*(v)$), then u is in the kernel of T. Using this and the last part, show that $\text{kernel}(T) = \text{image}(T^*)^\perp$.

(iii) Derive the identities in (1) from the result of the last part using the relations $(W^\perp)^\perp = W$ and $T^{**} = T$.

6. Suppose V is a vector space over complex numbers $\mathbb{C}$ with an inner product $\langle\, , \rangle : V \times V \to \mathbb{C}$. Since $\mathbb{R} \subset \mathbb{C}$, the vector space V can be also regarded as a vector space over real numbers $\mathbb{R}$. For any two vector $u, v \in V$, let $[u,v] \in \mathbb{R}$ be the real part of $\langle u,v \rangle \in \mathbb{C}$. Show that if V is regarded as a vector space over $\mathbb{R}$, then $[\,,]$ defines an inner product on V. Show also that $[v, \sqrt{-1}v] = 0$.