Problem Set 1

1. *Define addition and scalar multiplication on $\mathbb{R}^3$ as follows:*

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{a_1} + \sqrt[3]{b_1})^3 \\ (\sqrt[3]{a_2} + \sqrt[3]{b_2})^3 \\ (\sqrt[3]{a_3} + \sqrt[3]{b_3})^3 \end{bmatrix}, \qquad \lambda \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \lambda^3 a_1 \\ \lambda^3 a_2 \\ \lambda^3 a_3 \end{bmatrix}.$$

Show that $\mathbb{R}^3$ together with these operations defines a vector space.

Solution. *The addition is symmetric:*

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{a_1} + \sqrt[3]{b_1})^3 \\ (\sqrt[3]{a_2} + \sqrt[3]{b_2})^3 \\ (\sqrt[3]{a_3} + \sqrt[3]{b_3})^3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{b_1} + \sqrt[3]{a_1})^3 \\ (\sqrt[3]{b_2} + \sqrt[3]{a_2})^3 \\ (\sqrt[3]{b_3} + \sqrt[3]{a_3})^3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and associative:

$$\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{a_1} + \sqrt[3]{b_1})^3 \\ (\sqrt[3]{a_2} + \sqrt[3]{b_2})^3 \\ (\sqrt[3]{a_3} + \sqrt[3]{b_3})^3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{a_1} + \sqrt[3]{b_1} + \sqrt[3]{c_1})^3 \\ (\sqrt[3]{a_2} + \sqrt[3]{b_2} + \sqrt[3]{c_2})^3 \\ (\sqrt[3]{a_3} + \sqrt[3]{b_3} + \sqrt[3]{c_1})^3 \end{bmatrix} =$$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} (\sqrt[3]{b_1} + \sqrt[3]{c_1})^3 \\ (\sqrt[3]{b_2} + \sqrt[3]{c_2})^3 \\ (\sqrt[3]{b_3} + \sqrt[3]{c_3})^3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \right).$$

The vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ *is the neutral element for addiction and the additive inverse of* $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *is equal to* $\begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}$. *Scaling of any vector* $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ *by $\lambda = 1$ is clearly equal to* $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. *We also have:*

$$\mu \left(\lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) = \mu \begin{bmatrix} \lambda^3 a_1 \\ \lambda^3 a_2 \\ \lambda^3 a_3 \end{bmatrix} = \begin{bmatrix} \mu^3 \lambda^3 a_1 \\ \mu^3 \lambda^3 a_2 \\ \mu^3 \lambda^3 a_3 \end{bmatrix} = (\mu\lambda) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

$$(\mu+\lambda) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} (\mu+\lambda)^3 a_1 \\ (\mu+\lambda)^3 a_2 \\ (\mu+\lambda)^3 a_3 \end{bmatrix} = \begin{bmatrix} (\mu\sqrt[3]{a_1} + \lambda\sqrt[3]{a_1})^3 \\ (\mu\sqrt[3]{a_2} + \lambda\sqrt[3]{a_2})^3 \\ (\mu\sqrt[3]{a_3} + \lambda\sqrt[3]{a_3})^3 \end{bmatrix} = \begin{bmatrix} \mu^3 a_1 \\ \mu^3 a_2 \\ \mu^3 a_3 \end{bmatrix} + \begin{bmatrix} \lambda^3 a_1 \\ \lambda^3 a_2 \\ \lambda^3 a_3 \end{bmatrix} = \mu \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

and:

$$\lambda \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = \lambda \begin{bmatrix} (\sqrt[3]{a_1} + \sqrt[3]{b_1})^3 \\ (\sqrt[3]{a_2} + \sqrt[3]{b_2})^3 \\ (\sqrt[3]{a_3} + \sqrt[3]{b_3})^3 \end{bmatrix} = \begin{bmatrix} \lambda^3(\sqrt[3]{a_1} + \sqrt[3]{b_1})^3 \\ \lambda^3(\sqrt[3]{a_2} + \sqrt[3]{b_2})^3 \\ \lambda^3(\sqrt[3]{a_3} + \sqrt[3]{b_3})^3 \end{bmatrix} = \begin{bmatrix} (\sqrt[3]{\lambda^3 a_1} + \sqrt[3]{\lambda^3 b_1})^3 \\ (\sqrt[3]{\lambda^3 a_2} + \sqrt[3]{\lambda^3 b_2})^3 \\ (\sqrt[3]{\lambda^3 a_3} + \sqrt[3]{\lambda^3 b_3})^3 \end{bmatrix} =$$

$$\begin{bmatrix} \lambda^3 a_1 \\ \lambda^3 a_2 \\ \lambda^3 a_3 \end{bmatrix} + \begin{bmatrix} \lambda^3 b_1 \\ \lambda^3 b_2 \\ \lambda^3 b_3 \end{bmatrix} = \lambda \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This complete the proof of showing that $\mathbb{R}^3$ with the above addition and scalar multiplication is a vector space.

2. *For a vector x in a vector space V and a scalar $\lambda \in \mathbb{R}$, let $\lambda v = 0$. Show that either $\lambda = 0$ or $v = 0$.*

Solution. *If $\lambda = 0$, then we are done. If $\lambda \neq 0$, then we have:*

$$0 = \frac{1}{\lambda}(\lambda v) = (\frac{1}{\lambda} \cdot \lambda)v = 1v = v$$

Therefore, $v = 0$.

3. *Are the following subsets of* $C(\mathbb{R},\mathbb{R})$ *subspaces? Justify your answer in each case.*

(i) $W_1 := \{f : \mathbb{R} \to \mathbb{R} \mid f$ *is continuous and* $f(1) = 0\}$.

Solution. *Suppose* f *and* g *belong to* W_1 *and* $c \in \mathbb{R}$. *Then we have:*

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0$$

and

$$(cf)(1) = c \times f(1) = c \times 0 = 0.$$

(ii) $W_2 := \{f : \mathbb{R} \to \mathbb{R} \mid f$ *is continuous and* $f(0) = 1\}$.

Solution. *This set is not a subspace of* $C(\mathbb{R},\mathbb{R})$ *because the constant function* $f(x) \equiv 1$ *is in* W_2, *but* $0 \cdot f$ *is not in* W_2.

4. *The transpose of a matrix* $A = [a_{i,j}]$ *in* $M_{m\times n}(\mathbb{R})$ *is the matrix* $A^t = \left[a^t_{i,j}\right]$ *in* $M_{n\times m}(\mathbb{R})$ *such that*

$$a^t_{i,j} = a_{j,i}.$$

A matrix $A \in M_{n\times n}(\mathbb{R})$ *is anti-symmetric if* $A = -A^t$.

(i) *Write down the general form of an anti-symmetric matrix in* $M_{3\times 3}(\mathbb{R})$.

Solution.

$$\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

(ii) *Show that the set of all anti-symmetric matrices is a subspace of* $M_{n\times n}(\mathbb{R})$.

Solution. *Suppose* A, B *are anti-symmetric matrices in* $M_{n\times n}(\mathbb{R})$ *and* $c \in \mathbb{R}$. *Then we have:*

$$(A+B)^t = A^t + B^t = (-A) + (-B) = -(A+B)$$

and

$$(cA)^t = cA^t = -cA.$$

Therefore, the set of anti-symmetric matrices forms a subspace of $M_{n\times n}(\mathbb{R})$.

5. *A field is a set F together with the maps $+ : F \times F \to F$ and $\cdot : F \times F \to F$ which satisfy the following properties:*

(F1) $a + b = b + a$ *for any* $a, b \in F$*;*

(F2) $(a + b) + c = a + (b + c)$ *for any* $a, b, c \in F$*;*

(F3) There is $0 \in F$ *such that for* $a \in F$*, we have* $a + 0 = a$*;*

(F4) For any $a \in F$*, there is* $a' \in F$ *such that* $a + a' = 0$*;*

(F5) $a \cdot b = b \cdot a$ *for any* $a, b \in F$*;*

(F6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ *for any* $a, b, c \in F$*;*

(F7) There is $1 \in F$ *such that for* $a \in F$*, we have* $a \cdot 1 = a$*;*

(F8) For any non-zero $a \in F$*, there is* $a'' \in F$ *such that* $a \cdot a'' = 1$*;*

(F9) $(a + b) \cdot c = a \cdot c + b \cdot c$ *for any* $a, b, c \in F$*.*

(i) Let S be a set with two elements. Define operations $+$ and $\cdot$ such that the resulting space is a field. We denote this field by $\mathbb{Z}/2$. (You do not need to write down the verification of the above properties.)

Solution. *We denote the two elements of $\mathbb{Z}/2$ by 0 (neutral element for addition) and 1 (neutral element for multiplication). We define addition and multiplication on $\mathbb{Z}/2$ as follows:*

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

(ii) We may define the notion of a vector space V over a field F in the same way as before, except that scalar multiplcation is defined as the multiplication of an element of F and a vector V. In particular, the vector addition and scalar multiplication need to satisfy the eight properties that we listed in the class. Let $M_{m\times n}(F)$ be the space of m by n matrices such that each entry is an element of F. Define a vector space structure on $M_{m\times n}(F)$ generalizing the definition of addition and scalar multiplication for $M_{m\times n}(\mathbb{R})$.

Solution. *Let $A = [a_{i,j}]_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ and $B = [b_{i,j}]_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}$ are two elements of $M_{m\times n}(F)$ and $c \in F$. then we define:*

$$A + B = [a_{i,j} + b_{i,j}]_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}, \qquad cA = [ca_{i,j}]_{\substack{1\leq i\leq m\\ 1\leq j\leq n}}.$$

It can be easily seen that this addition and scalar multiplication define a vector space structure on $M_{m\times n}(F)$.

(iii) How many elements does the vector space $M_{m\times n}(\mathbb{Z}/2)$ have?

Solution. *An element of $M_{m\times n}(\mathbb{Z}/2)$ has mn entries and there are two possibilities for each entry. Therefore, there are 2^{mn} matrices in $M_{m\times n}(\mathbb{Z}/2)$.*