Problem Set 2

1. In each of the following cases determine whether the given set is a basis for $\mathbb{R}^n$:

(i) $n = 3$ and $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$

(ii) $n = 4$ and $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$

(iii) $n = 3$ and $S = \left\{ \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix} \right\}.$

2. In each of the following cases determine whether the given matrices form a basis for $M_{2\times 2}(\mathbb{R})$:

(i) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$

(ii) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$

3. Suppose S is a linearly independent generating set for a vector space V. Show that S is an efficient generating set, *i.e.*, any proper subset of S is not a generating set.

4. Write down three different bases for the vector space $M_{2\times 1}(\mathbb{Z}/2\mathbb{Z})$. (Recall that $M_{m\times n}(\mathbb{Z}/2\mathbb{Z})$ is defined in Problem 5 of the first problem set.)

5. Let $\mathcal{P}$ denote the subspace of $C(\mathbb{R},\mathbb{R})$ defined as follows:

$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a polynomial}\}.$$

Recall that a polynomial $f(x)$ is a function which has the following form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

for some choice of n, a_0, a_1, …, a_n. Find a basis for $\mathcal{P}$.

6. Suppose $\{x_1,x_2,\ldots,x_n\}$ is a basis for a vector space V. Show that each of the following sets is also a basis for V.

   (i) $\{x_1 + x_2, x_2, x_3, \ldots, x_n\}$.

   (i) $\{\lambda \cdot x_1, x_2, x_3, \ldots, x_n\}$, where λ is a non-zero number.