Problem Set 2

1. *In each of the following cases determine whether the given set is a basis for* $\mathbb{R}^n$*:*

   (i) $n = 3$ *and* $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$

   Solution. *This set is not a basis because these vectors are linearly dependent:*

$$2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   (ii) $n = 4$ *and* $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$

   Solution. *Because the dimension of* $\mathbb{R}^4$ *is four, and there are four elements in* S*, we just need to show that these vectors are linearly independent.*

$$x \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + w \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{cases} x + 3z + 4w = 0 \\ y - 2z = 0 \\ x + w = 0 \\ y + z = 0 \end{cases} \implies \begin{cases} 3z + 3w = 0 \\ y - 2z = 0 \\ x + w = 0 \\ 3z = 0 \end{cases}$$

   To obtain the second system from the first system, we subtract the third equation from the first one and subtract the second equation from the fourth equation. The last equation of the second system implies that $z = 0$*. Now from the first equation we conclude that* $w = 0$*. Consequently, we can obtain from the second and the third equations that* $y = 0$ *and* $x = 0$*. Therefore the given four vectors are linearly independent and* S *is a basis.*

   (iii) $n = 3$ *and* $S = \left\{ \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix} \right\}.$

   Solution. *The dimension of* $\mathbb{R}^3$ *is equal to 3. Thus* S *cannot be a basis for* $\mathbb{R}^3$ *because it has four elements.*

2. *In each of the following cases determine whether the given matrices form a basis for* $M_{2\times 2}(\mathbb{R})$*:*

(i) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$

Solution. *This set is not a basis because these vectors are linearly dependent:*

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(ii) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$

Solution. *We firstly show that these matrices are linearly independent.*

$$x\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + w\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \begin{cases} x+y=0 \\ x+w=0 \\ y+z=0 \\ z-w=0 \end{cases} \implies$$

$$\implies \begin{cases} x+y=0 \\ y-w=0 \\ y+z=0 \\ z-w=0 \end{cases} \implies \begin{cases} x+y=0 \\ y-w=0 \\ z+w=0 \\ z-w=0 \end{cases}$$

The last two equations of the last system imply that $z = w = 0$*. Thus the second equation implies that* $y = 0$ *and then we use the first equation to see that* $x = 0$*. This shows that the four matrices are linearly independent. Also note that:*

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \left(\frac{a+b-c+d}{2}\right)\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \left(\frac{a-b+c-d}{2}\right)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \left(\frac{-a+b+c+d}{2}\right)\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$
$$+ \left(\frac{-a+b+c-d}{2}\right)\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

which shows that the four given matrices are generating. So they determine a basis.

3. *Suppose S is a linearly independent generating set for a vector space V. Show that S is an efficient generating set, i.e., any proper subset of S is not a generating set.*

Solution. *Suppose x_0 is an element of S such that $S \setminus \{x_0\}$ is still a basis. Then x_0 can be written as linear combination of the elements of $S \setminus \{x_0\}$ which means that there are $x_1, \dots, x_k \in S \setminus \{x_0\}$ and the scalars $a_1, \dots, a_k$ such that*

$$x_0 = a_1x_1 + a_2x_2 + \cdots + a_kx_k \implies -x_0 + a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$$

This means that S is linearly dependent which is a contradiction. Thus S is an efficient generating set.

4. *Write down three different bases for the vector space $M_{2\times 1}(\mathbb{Z}/2\mathbb{Z})$. (Recall that $M_{m\times n}(\mathbb{Z}/2\mathbb{Z})$ is defined in Problem 5 of the first problem set.)*

Solution. *As the first basis, we consider the standard one* $S_1 = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. *In particular, the dimension of $M_{2\times 1}(\mathbb{Z}/2\mathbb{Z})$ is equal to 2. As the other two bases, we consider*

$$S_2 = \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \qquad S_3 = \left\{ f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Notice that S_2 and S_3, they both have $2 = \dim(M_{2\times 1}(\mathbb{Z}/2\mathbb{Z}))$ elements and to show that they are bases, we just need to show that they are generating. To see this, take an arbitrary element $\begin{bmatrix} a \\ b \end{bmatrix} \in M_{2\times 1}(\mathbb{Z}/2\mathbb{Z})$:

$$\begin{bmatrix} a \\ b \end{bmatrix} = (a+b)e_1 + bf = af + (a+b)e_2.$$

This shows that S_2 and S_3 are generating and hence they are bases for $M_{2\times 1}(\mathbb{Z}/2\mathbb{Z})$.

5. *Let $\mathcal{P}$ denote the subspace of $C(\mathbb{R},\mathbb{R})$ defined as follows:*

$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a polynomial}\}.$$

Recall that a polynomial $f(x)$ is a function which has the following form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for some choice of n, a_0, a_1, ..., a_n. Find a basis for $\mathcal{P}$.

Solution. *The set $S = \{1, x, x^2, \dots\}$ gives a basis for $\mathcal{P}$ which has infinitely many elements. From the definition of $\mathcal{P}$, it is clear that any polynomial can be written as a finite linear combination of the elements of S. Moreover, if a linear combination of the elements of S is zero, it means that there are constants a_0, a_1, ... such that a_i is zero if i is large enough and*

$$a_0 + a_1 x_1 + a_2 x_2 + \cdots = 0. \tag{1}$$

We pick N such that $a_i = 0$ if $i > N$. Then the above linear combination can be written as

$$P(x) = a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_N x_N = 0.$$

Since $P(x) = 0$, it implies that $P(0) = 0$, which in turn means that $a_0 = 0$. More generally, $P(x) = 0$ implies that the i^{th} derivative of $P(x)$ vanishes at any point including 0. It is easy to see that $\frac{d^i P}{dx^i}(0) = a_i$. Therefore, $a_i = 0$ for any i, which means that all coefficients in the linear combination (1) are equal to zero. This implies that S is linearly independent and hence it is a basis.

6. *Suppose $S_0 = \{x_1,x_2,\ldots,x_n\}$ is a basis for a vector space V. Show that each of the following sets is also a basis for V.*

   (i) $S_1 = \{x_1 + x_2, x_2, x_3, \ldots, x_n\}$.

   Solution. *Since S_0 is a basis for V, the dimension of V is n. So to show that S_1 is a basis for V, it suffices to show that it is a generating set. Suppose v is an arbitrary vector in V. Since S_0 is a basis for V, we can find scalars a_1, a_2, $\ldots$, a_n such that*

$$\begin{aligned} v &= a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \\ &= a_1(x_1 + x_2) + (a_2 - a_1)x_2 + a_3x_3 + \cdots + a_nx_n \end{aligned}$$

   This shows that any vector can be written as a linear combination of S_1.

   (i) $S_2 = \{\lambda \cdot x_1, x_2, x_3, \ldots, x_n\}$, *where λ is a non-zero number.*

Solution. *To show that S_2 is a basis for V, it suffices to show that it is a generating set because it has $n = \dim(V)$ elements. Suppose v is an arbitrary vector in V. Since S_0 is a basis for V, we can find scalars a_1, a_2, $\ldots$, a_n such that*

$$\begin{aligned} v &= a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \\ &= \frac{a_1}{\lambda}(\lambda x_1) + a_2x_2 + a_3x_3 + \cdots + a_nx_n \end{aligned}$$

This shows that any vector can be written as a linear combination of S_2.