# Problem Set 3

1. Determine all linear subspaces of the vector space $\mathbb{R}^3$.

2. Determine the kernel of the following linear transformations:

(i) $$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y + 6z \\ 5x + y \\ 7x + y - 4z \end{bmatrix}.$$

(ii) $$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 3x + 6y + 5z - 6w \\ x + 2z + 7w + 8y \end{bmatrix}.$$

3. Verify whether the vector $v$ is in the image of of the linear transformation $T(x) = Ax$.

(i) $v = \begin{bmatrix} 8 \\ 3 \\ 1 \\ 4 \end{bmatrix}, A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$

(ii) $v = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 2 & -3 & 6 \\ -8 & 1 & -6 \\ 3 & 1 & 0 \end{bmatrix}.$

4. (i) Give an example of a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that the kernel and the image of $T$ are equal to each other.

(ii) Find linear maps $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that they have the same kernels and the same images but $T_2$ is not a multiple of $T_1$.

5. Suppose $W_1$ and $W_2$ are subspaces of a vector space $V$. We define $W_1 + W_2$ to be the subset of $V$ consisting of vectors $x$ in $V$ which can be written as $x_1 + x_2$ where $x_i \in W_i$.

(i) Show that $W_1 + W_2$ is a subspace of $V$.

(ii) Show that the dimension of $W_1 + W_2$ is at most equal to $\dim(W_1) + \dim(W_2)$.

(iii) By giving an example, show that it is possible that the dimension of $W_1 + W_2$ is strictly less than $\dim(W_1) + \dim(W_2)$.

6. (i) Suppose $V$ is a finite dimensional vector space and $T : V \to V$ is a linear transformation with trivial kernel. Show that $T$ is surjective.

(ii) Suppose $T : C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ is the linear transformation given by

$$T(f)(x) := \int_0^x f(t)\, dt.$$

In the class we saw $T$ as an example of a linear map. Show that $T$ has trivial kernel. By determining the image of this map show that the claim in part (i) is not correct anymore if we drop the finite dimensionality assumption on $V$.