Problem Set 3

1. *Determine all linear subspaces of the vector space* $\mathbb{R}^3$.

Solution. *Possible dimensions for a subspace* W *of* $\mathbb{R}^3$ *are* 0, 1, 2 *and* 3.

Case 1 ($\dim(V) = 0$) *The only* 0*-dimensional subspace of a vector space is the vector space* $\{0\}$.

Case 2 ($\dim(V) = 1$) *A* 1*-dimensional subspace of* $\mathbb{R}^3$ *can be written as*

$$\left\{ \begin{bmatrix} at \\ bt \\ ct \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

where $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *is a fixed non-zero vector in* $\mathbb{R}^3$. *Geometrically, this is a line which passes through the origin. Varying the non-zero vector* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *exhausts all possible 1-dimensional subspaces of* $\mathbb{R}^3$.

Case 3 ($\dim(V) = 2$) *A* 2*-dimensional subspace of* $\mathbb{R}^3$ *can be written as*

$$\left\{ \begin{bmatrix} at + a's \\ bt + b's \\ ct + c's \end{bmatrix} \mid t,s \in \mathbb{R} \right\}$$

where $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *and* $\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ *are two fixed vectors in* $\mathbb{R}^3$ *which are not linearly independent. Geometrically, this is a plane which passes through the origin. Varying the linearly independent vectors* $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ *and* $\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ *exhausts all possible 2-dimensional subspaces of* $\mathbb{R}^3$.

Case 3 ($\dim(V) = 3$) *Any* n*-dimensional subspace of an* n*-dimensional vector space* X *is equal to* X. *In particular, any* 3*-dimensional subspace of* $\mathbb{R}^3$ *is equal to* $\mathbb{R}^3$.

Solution. *Take* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *to be the linear map given by the matrix*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then the image and the kernel of T *can be both identified with the line*

$$L = \{ \begin{bmatrix} t \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \}.$$

2. *Determine the kernel of the following linear transformations:*

(i) $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x+y+6z \\ 5x+y \\ 7x+y-4z \end{bmatrix}.$

Solution. *Suppose* $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *is in the kernel of* T*. Then we have:*

$$\begin{cases} 2x+y+6z=0 \\ 5x+y=0 \\ 7x+y-4z=0 \end{cases} \implies \begin{cases} 2x+y+6z=0 \\ 10x+2y=0 \\ 14x+2y-8z=0 \end{cases} \implies \begin{cases} 2x+y+6z=0 \\ (-3)y-(30)z=0 \\ (-5)y-(50)z=0 \end{cases} \implies \begin{cases} 2x+y+6z=0 \\ y+10z=0 \\ y+10z=0 \end{cases}$$

The second system is obtained from the first one by multiplying the second and the third equations by 2*. Then subtracting multiples of the first equation from the second and the third equation gives rise to the third system. By dividing the second and the third equations of the third system by* (-3) *and* (-5)*, we obtain the last system. Now we subtract the second equation from the first and the third equations to obtain the following system:*

$$\begin{cases} 2x-4z=0 \\ y+10z=0 \\ 0=0 \end{cases}$$

Now if we pick z *arbitrarily, then the first equation can be used to determine* x*, the second equation can be used to determine* y *and the third equation is always satisfied. Therefore, a general solution to the above system is given as:*

$$\left\{\begin{bmatrix} 2t \\ -10t \\ t \end{bmatrix} \mid t \in \mathbb{R}\right\}$$

(ii) $T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 3x+6y+5z-6w \\ x+2z+7w+8y \end{bmatrix}.$

Solution. *Suppose* $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ *is in the kernel of* T*. Then we have:*

$$\begin{cases} 3x+6y+5z-6w=0 \\ x+8y+2z+7w=0 \end{cases} \implies \begin{cases} 18y-z-27w=0 \\ x+8y+2z+7w=0 \end{cases} \implies \begin{cases} 18y-z-27w=0 \\ x+44y-47w=0 \end{cases}$$

The second system is obtained from the first by subtracting third times the second equation from the first equation. Then adding twice of the first equation to the second one gives the third system. Now if we pick w *and* y *arbitrarily, then the second equation can be used to determine* x *and the first equation can be used to determine* z*. Therefore, a general solution to the above system is given as:*

$$\left\{\begin{bmatrix} 47t-44s \\ s \\ 18s-27t \\ t \end{bmatrix} \mid s,t \in \mathbb{R}\right\}.$$

3. *Verify whether the vector v is in the image of of the linear transformation $T(x) = Ax$.*

(i) $v = \begin{bmatrix} 8 \\ 3 \\ 1 \\ 4 \end{bmatrix}, A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$

Solution. *If the vector v is equal to $A\begin{bmatrix} x \\ y \end{bmatrix}$, then x and y need to satisfy the following*

$$x\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 1 \\ 4 \end{bmatrix} \implies \begin{cases} 2x + 4y = 8 \\ 2y = 3 \\ x = 1 \\ x + 2y = 4 \end{cases}$$

The above system clearly has the solution $x = 1$ and $y = \frac{3}{2}$. So v is in the image of T.

(ii) $v = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, A = \begin{bmatrix} 2 & -3 & 6 \\ -8 & 1 & -6 \\ 3 & 1 & 0 \end{bmatrix}.$

Solution. *If the vector v is equal to $A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, then x, y and z need to satisfy the following*

$$x\begin{bmatrix} 2 \\ -8 \\ 3 \end{bmatrix} + y\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 6 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \implies \begin{cases} 2x - 3y + 6z = 1 \\ (-8)x + y - 6z = 2 \\ 3x + y = 5 \end{cases}$$

We add the first equation and twice of the third equation to the second equation to obtain the following system

$$\begin{cases} 2x - 3y + 6z = 1 \\ 0 = 13 \\ 3x + y = 5 \end{cases}$$

This system does not have any solution because the second equation is never satisfied. So v is not in the image of T.

4. *(i) Give an example of a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that the kernel and the image of T are equal to each other.*

Solution. *Take $T : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear map given by the matrix*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then the image and the kernel of T can be both identified with the line

$$L = \{ \begin{bmatrix} t \\ 0 \end{bmatrix} \mid t \in \mathbb{R} \}.$$

(ii) Find linear maps $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that they have the same kernels and the same images but T_2 is not a multiple of T_1.

Solution. *Take $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ to be the linear maps given by the matrices*

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

It is straightforward to check that the images of T_1 and T_2 are both equal to $\mathbb{R}^2$, and their kernels are equal to the zero vector space. However, T_1 and T_2 are not multiple of each other.

5. *Suppose W_1 and W_2 are subspaces of a vector space V. We define $W_1 + W_2$ to be the subset of V consisting of vectors x in V which can be written as $x_1 + x_2$ where $x_i \in W_i$.*

   (i) *Show that $W_1 + W_2$ is a subspace of V.*

   Solution. *Firstly note that 0 is clearly an element of $W_1 + W_2$ because $0 = 0 + 0$. Suppose $v, v' \in W_1 + W_2$ and $c \in \mathbb{R}$. Therefore, there are $x_1, x_1' \in W_1$ and $x_2, x_2' \in W_2$ such that*

$$v = x_1 + x_2, \qquad v' = x_1' + x_2'.$$

   Then we have

$$v + v' = \underbrace{(x_1 + x_1')}_{\in W_1} + \underbrace{(x_2 + x_2')}_{\in W_2}, \qquad\qquad cv = \underbrace{(cx_1)}_{\in W_1} + \underbrace{(cx_2)}_{\in W_2}$$

   which completes the proof of the claim that $W_1 + W_2$ is a subspace.

   (ii) *Show that the dimension of $W_1 + W_2$ is at most equal to $\dim(W_1) + \dim(W_2)$.*

   Solution. *Suppose $S_1 = \{x_1, \ldots, x_k\}$ is a basis for W_1 and $S_2 = \{y_1, \ldots, y_l\}$ is a basis for W_2. This means that:*

$$k = \dim(W_1), \qquad l = \dim(W_2).$$

   To prove the claim, we show that $S_1 \cup S_2$ is a generating set for $W_1 + W_2$. This would show that $\dim(W_1 + W_2) \leq k + l$ which completes the proof.

   Suppose v is a vector in $W_1 + W_2$. Then it can be written as $z_1 + z_2$ where $z_i \in W_i$. Since S_i is a basis (and hence a generating set) for W_i, we can write z_i as a linear combination of the elements of S_i

$$z_1 = a_1x_1 + a_2x_2 + \cdots + a_kx_k, \qquad z_2 = a_1x_1 + a_2x_2 + \cdots + a_kx_k.$$

   Thus we have

$$v = a_1x_1 + a_2x_2 + \cdots + a_kx_k + a_1x_1 + a_2x_2 + \cdots + a_kx_k.$$

   So $v \in \mathrm{Span}(S_1 \cup S_2)$.

   (iii) *By giving an example, show that it is possible that the dimension of $W_1 + W_2$ is strictly less than $\dim(W_1) + \dim(W_2)$.*

   Solution. *Take $V = \mathbb{R}^2$, $W_1 = W_2 = \mathbb{R}^2$. Then $W_1 + W_2 = \mathbb{R}^2$. We also have:*

$$2 = \dim(W_1 + W_2) < \dim(W_1) + \dim(W_2) = 4.$$

6. *(i) Suppose V is a finite dimensional vector space and $T : V \to V$ is a linear transformation with trivial kernel. Show that T is surjective.*

Solution. *Suppose $T : V \to V$ is an injective linear transformation with $\dim(V)$ being finite. Then dimension formula implies that*

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V).$$

Since T is injective, $\text{Ker}(T)$ is trivial and hence $\text{Nullity}(T) = 0$. Using this fact, we can rewrite the above equality as

$$\text{Rank}(T) = \dim(V).$$

Therefore, the dimension of the image of T, which is a subspace of V, is equal to $\dim(V)$. This implies that the image of T is equal to V because V is finite dimensional. Consequently, T is surjective.

(ii) Suppose $T : C(\mathbb{R},\mathbb{R}) \to C(\mathbb{R},\mathbb{R})$ is the linear transformation given by

$$T(f)(x) := \int_0^x f(t)\,dt.$$

In the class we saw T as an example of a linear map. Show that T has trivial kernel. By determining the image of this map show that the claim in part (i) is not correct anymore if we drop the finite dimensionality assumption on V.

Solution. *The fundamental theorem of calculus implies that for any $f \in C(\mathbb{R},\mathbb{R})$, the function $T(f) : \mathbb{R} \to \mathbb{R}$ is differentiable and its derivative is equal to f. If $f \in \ker(T)$, then $T(f) = 0$ and hence its derivative, which is equal to f, is also the zero function. Therefore, $\ker(T)$ contains only the zero function and T is injective. On the other hand, T is not surjective because any function in the image of T is differentiable and there are continuous functions which are not differentiable. For example, the absolute value function $g(x) = |x|$ is one such function.*