# Problem Set 4

1. (i) Compute $A(B+C)D$ where:

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}.$$

(ii) Compute $A^t B$ where:

$$A = \begin{bmatrix} -1 & 0 \\ 7 & -3 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 6 \\ -2 & 1 & -3 \\ -1 & 1 & 8 \end{bmatrix}.$$

2. Suppose $\mathcal{P}_n$ denotes the space of all polynomials of degree at most $n$. That is to say any element of $\mathcal{P}_n$ can be written as:

$$a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0$, $a_1$, $\ldots$, $a_n \in \mathbb{R}$. Then $\mathcal{P}_n$ is a vector space with basis $S_n = \{1, x, \ldots, x^n\}$. (You don't need to prove this.) Let $T : \mathcal{P}_3 \to \mathcal{P}_4$ be the map given by

$$T(p) := \frac{dp}{dx} - 2x \cdot p$$

where $p \in \mathcal{P}_3$.

(i) Show that $T : \mathcal{P}_3 \to \mathcal{P}_4$ is a linear map.

(ii) Find the matrix representation $[T]_{S_3}^{S_4}$ of the linear transformation $T$.

3. Consider the following linear transformation $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$:

$$T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(i) Find the matrix $[T]_S^S$ where $S$ is the following basis of $M_{2\times 2}(\mathbb{R})$:

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(ii) Find bases for the image and kernel of $T$ and determine the rank and nullity of $T$.

4. Suppose $V$ is a finite dimensional vector space and $T : V \to V$ is a linear transformation. Show that there are bases $S$ and $S'$ for $V$ such that the matrix $[T]_S^{S'}$ is a diagonal matrix, namely, it has non-zero terms only on the diagonal entries.