Problem Set 5

1. In each part, let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $T_A(x) = Ax$, defined using the matrix $M_{n\times n}(\mathbb{R})$. Let also S be the given oriented basis for $\mathbb{R}^n$. (You don't need to show that S is a basis.) Find the matrix representation $[T_A]_S^S$.

   (i) $n = 2$, $A = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$ and $S = \{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$.

   (ii) $n = 3$, $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ and $S = \{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\}$.

2. In each part, find the reduced row echelon form of the given matrix:

(i) $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 5 & -1 & 6 \\ 2 & 4 & 1 & 2 \\ 2 & 0 & -7 & 11 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 2 & -2 & -1 & 6 & -2 \\ 1 & -1 & 1 & 2 & -1 \\ 4 & -4 & 5 & 7 & -1 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 2 & 0 & 3 & 0 & -4 \\ 3 & -4 & 8 & 3 & 0 \\ 1 & -1 & 2 & 1 & -1 \\ -2 & 5 & -9 & -3 & -5 \end{bmatrix}$

3. Solve the equation $A\vec{x} = \vec{b}$. Notice that in each part A is in reduced row echelon form.

(i) $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 0 & 7 & -3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 6 & -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 0 & 2 & 8 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 4 \\ 3 \\ 4 \\ 4 \end{bmatrix}$

4. Suppose we have the following information about the matrix $A \in M_{4\times 5}$:

(i) The reduced row echelon form of A is the following matrix:

$$\begin{bmatrix} 1 & 0 & 10 & 0 & 1 \\ 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) The first, the second and the fourth columns of A are respectively equal to:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}.$$

Find the matrix A.

5. (i) Find a 2×2 matrix A and vectors $\vec{b}, \vec{b'} \in \mathbb{R}^2$ such that the system $A\vec{x} = \vec{b}$ doesn't have any solution and the system $A\vec{x} = \vec{b'}$ has infinitely many solutions.

(ii) Find an example of a homogenous system with three variables and three equations which has infinitely many solutions.

(iii) Is it possible to find a 3×3 matrix A and vectors $\vec{b}, \vec{b'} \in \mathbb{R}^3$ such that the system $A\vec{x} = \vec{b}$ has a unique solution and the system $A\vec{x} = \vec{b'}$ has infinitely many solutions. (Hint: Notice that you only need to consider the case that A is in reduced row echelon form.)