Problem Set 6

1. In each part, determine the tank of the given matrix:

(i) $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 3 & 3 & -1 & 6 \\ 4 & 5 & -1 & 8 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 \\ 1 & 2 & -1 & 1 \end{bmatrix}$

2. For a matrix $A \in M_{m\times n}(\mathbb{R})$, let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation $T_A(x) = Ax$. For each of the matrices in Problem 1, describe the vectors in the image of T_A as the set of solutions to a few linear equations.

3. Determine whether the given matrix A is invertible, and if it is invertible find its inverse.

(i) $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & -1 \\ 2 & 4 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 2 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 8 & 7 & 2 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 5 & -1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

4. Suppose A is an $n \times n$ matrix. As usual, the linear transformation associated to A is denoted by $T_A : \mathbb{R}^n \to \mathbb{R}^n$.

   (i) Suppose there is a matrix B such that $BA = I_n$. Show that the linear transformation T_A has trivial kernel.

   (ii) Now suppose that there is a matrix C such that such that $AC = I_n$. Show that the linear transformation T_A has tank n.

   (iii) Use the first part to show that if for $A \in M_{n \times n}(\mathbb{R})$ there is a matrix B such that $BA = I_n$, then A is invertible. (Recall that to show that A is invertible, it suffices to show that T_A is invertible.)

   (iv) Use the second part to show that if for $A \in M_{n \times n}(\mathbb{R})$ there is a matrix C such that $AC = I_n$, then A is invertible.

5. Show that any invertible matrix can be written as a product of elementary matrices. (Hint: Use the fact that the reduce row echelon form of an invertible matrix is an identity matrix.)

6. For an invertible transformation $T : V \to W$, show that for any subspace X of V, the dimension of $T(X)$ and X are equal to each other.

7. Suppose $A \in M_{m\times n}(\mathbb{R})$ is a matrix with rank m. Show that there is an $n \times m$ matrix B such that $AB = I_m$. (Hint: Try to determine columns of B one by one.)