Problem Set 7

1. Compute the determinant of the given matrix. In each part, give all the steps for your final answer.

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 3 & -1 & 6 \\ 2 & 4 & -1 & -5 \\ 1 & 5 & -1 & 9 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{bmatrix}$

2. Determine the determinant of the following matrix. Give the necessary steps to justify your answer.

$$\begin{bmatrix} x & 0 & 0 & 0 & a_0 \\ -1 & x & 0 & 0 & a_1 \\ 0 & -1 & x & 0 & a_2 \\ 0 & 0 & -1 & x & a_3 \\ 0 & 0 & 0 & -1 & x + a_4 \end{bmatrix}$$

3. In each part, find k such that the given identity holds for all values of $a_1, a_2, \ldots, c_3$.

(i) $$\det \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ 2c_1 & 2c_2 & 2c_3 \end{bmatrix} = k \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

(ii) $$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ 2b_1 + a_1 & 2b_2 + a_2 & 2b_3 + a_3 \\ 3c_1 & 3c_2 & 3c_3 \end{bmatrix} = k \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

(iii) $$\det \begin{bmatrix} b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ a_1 + c_1 & a_2 + c_2 & a_3 + c_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{bmatrix} = k \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

4. (i) For each elementary matrix E determine the determinant of E.

- **E is the elementary matrix of type (I) corresponding to multiplication of the i^{th} row by the scalar c:**

- **E is the elementary matrix of type (II) corresponding to interchanging the i^{th} and the j^{th} rows:**

- **E is the elementary matrix of type (III) corresponding to adding c times the i^{th} row to the j^{th} row:**

(ii) For each of the three types of elementary matrices show that $\det(E^t) = \det(E)$.

5. We saw in the class that an $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. In this problem, we want to give a formula for the inverse of A assuming that $\det(A) \neq 0$. Suppose C is the $n \times n$ matrix whose entry in the i^{th} row and the j^{th} column is the cofactor $\widetilde{A}_{j,i}$ of A. (Note that the latter is $\widetilde{A}_{j,i}$, not $\widetilde{A}_{i,j}$).

   (i) In the case that $n = 2$ and
   $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$
   determine what C is.

   (ii) Show that any diagonal entry of the product matrix AC is equal to $\det(A)$. (Here A is a matrix of size n and is not necessarily a 2×2 matrix.)

   (iii) For $i \neq j$, let $d_{i,j}$ be the entry of the product AC in the i^{th} row and the j^{th} column. Show that $d_{i,j} = 0$. (Hint: To prove this, show that $d_{i,j}$ is equal to the determinant of the matrix obtained from A by replacing the j^{th} row with the i^{th} row. Then use the fact that a matrix with two equal rows has determinant zero.)

   (iv) Show that $B = \frac{1}{\det(A)} C$ is the inverse of A. (Hint: Recall from Problem 4 of the last homework that we just need to show that $AB = I_n$. Then $BA = I_n$ holds automatically and B is the inverse of A.)