Problem Set 8

1. For each part, find the eigenvalues and the corresponding eigenvectors of A. Determine whether each matrix is diagonalizable. If so, give the diagonal matrix.

   (i) $A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$

   (ii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$

   (iii) $A = \begin{bmatrix} 2 & -3 & 6 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. (i) Find all (real) eigenvalues of the following matrix. Is it possible to diagonalize this matrix? (This matrix gives the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which rotates a point by the angle t over the origin.)

$$\begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}$$

(ii) Find all (real) eigenvalues of the following matrix. Is it possible to diagonalize this matrix? (This matrix gives the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects a point with respect to an appropriate matrix.)

$$\begin{bmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{bmatrix}$$

3. (a) Determine the characteristic polynomial of the following matrix.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & 0 & a_1 \\ 0 & -1 & 0 & 0 & a_2 \\ 0 & 0 & -1 & 0 & a_3 \\ 0 & 0 & 0 & -1 & a_4 \end{bmatrix}$$

(b) Give explicit values for a_0, a_1, a_2, a_3 and a_4 such that the above matrix is diagonalizable. Explain why for your answer the matrix is diagonalizable.

4. (i) Show that for each square matrix A, $\det(A) = \det(A^t)$. (Hint: You can separate the problem into two cases that A is invertible and non-invertible. For an invertible matrix, recall that A can be written as the product of elementary matrices. Then use Problem 4 from the last problem set. In the case that A is non-invertible use the fact that $\text{Rank}(A^t) = \text{Rank}(A)$.)

(ii) Show that for each square matrix A, the characteristic polynomials of A and A^t are equal to each other.

5. Suppose V is the vector space $M_{3\times 3}(\mathbb{R})$ and $T : V \to V$ is the linear transformation which maps a matrix to its transpose.

   (i) Determine all eigenvalues of T.

   (ii) Determine all eigenvectors of T.