Problem Set 9

1. In each part, find the smallest T-invariant subspace W of V which contains the vector v:

(i) $V = \mathbb{R}^4$, $v = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ and T is given by multiplication by the matrix $A = \begin{bmatrix} 1 & 1 & 3 & 3 \\ -1 & 1 & 2 & 4 \\ 0 & -2 & 4 & 4 \\ 2 & 0 & 1 & -1 \end{bmatrix}$.

(ii) V is $\mathcal{P}_3$, the vector space of polynomials of degree at most 3, $v = x^3 + x$ and $T(p(x)) = p'(x)$.

(ii) $V = M_{2\times 2}(\mathbb{R})$, $v = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is given by $T(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} B$.

2. In each part, for the matrix A find a polynomial $p(x)$ such that $p(A) = 0$.

(i) $A = \begin{bmatrix} 2 & 2 & 6 & 1 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -11 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (Hint: Regard A as a rotation matrix, and then think about the geometric meaning of the powers A^k of A.)

3. Let A be a matrix and $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial such that $p(A) = 0$

(i) If $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, then show that $a_0 = a_1 = a_2 = a_3 = 0$. That is to say $p(x)$ can be written as $p(x) = x^4 q(x)$ for a polynomial $q(x)$.

(ii) If $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, then show that $p(1) = p(2) = p(3) = 0$. That is to say $p(x)$ can be written as $p(x) = (x-1)(x-2)(x-3)q(x)$ for a polynomial $q(x)$.

(iii) If $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, then show that $p(x)$ can be written as $p(x) = (x-2)^4 q(x)$ for a polynomial $q(x)$. (Hint: Any polynomial $p(x)$ can be written in the form $p(x) = a'_n(x-2)^n + a'_{n-1}(x-2)^{n-1} + \cdots + a'_1(x-2) + a'_0$. Use this presentation of $p(x)$ and then argue as in part (i).)

4. In each part, determine whether the pairing $\langle\,,\,\rangle$ determines an inner product on the vector space V. Justify your answer.

(i) $V = \mathbb{R}^2$, $\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x' \\ y' \end{bmatrix} \rangle = xx'$.

(ii) $V = \mathbb{R}^n$, $\langle \vec{v}, \vec{w} \rangle = (A\vec{v})^t A\vec{w}$ where t denotes transpose and A is an invertible matrix.

(iii) $V = \mathbb{R}^n$, $\langle \vec{v}, \vec{w} \rangle = (A\vec{v})^t A\vec{w}$ where t denotes transpose and A is an $n \times n$ non-invertible matrix.

(iv) $V = C([0,1],\mathbb{R})$, $\langle f(x), g(x) \rangle = \displaystyle\int_0^{\frac{1}{2}} f(x)g(x)dx$.

5. Suppose V is a vector space and $\langle\ ,\ \rangle$ is an inner product on V. Let $\vec{v}$ and $\vec{w}$ be non-zero vectors in V such that

$$\langle \vec{v}, \vec{w}\,\rangle = \|\vec{v}\| \cdot \|\vec{w}\|.$$

Show that $\vec{v}$ is a multiple of the vector $\vec{w}$. (Hint: Re-examine the proof of Cauchy-Schwarz inequality to see when the equality happens.)