Linear Algebra
Midterm 1 Spring 2020

Name: ______________________ **ID:** ______________________

Instructions:

(1) Fill in your name and Washington University ID at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. 1/2, not 2/4) to receive full credit.

(4) Read each questions carefully. Answer all questions in the space provided. If you need more room use the blank backs of the pages.

(5) Show your work; correct answers alone will receive only partial credit.

(6) This exam has 5 extra credit points.

Problem	1 (25 pts)	2 (30 pts)	3 (30 pts)	4 (20 pts)	Total (105 pts)
Score					

1. *In each of the following cases determine whether the given set is a basis for* $\mathbb{R}^n$*:*

(i) $n = 3$ *and* $S = \left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}.$

Solution. *This set is not a basis because these vectors are linearly dependent:*

$$2\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Grading: (10 points) *If you copied the relevant linear system incorrectly or made a minor arithmetic mistake, but followed a correct approach to conclude that the vectors are linearly independent, then you lose only* 1 *point.*

(ii) $n = 4$ *and* $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$

Solution. *Because the dimension of* $\mathbb{R}^4$ *is four, and there are four elements in* S*, we just need to show that these vectors are linearly independent.*

$$x\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} + w\begin{bmatrix} 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{cases} x + 3z + 4w = 0 \\ y - 2z = 0 \\ x + w = 0 \\ y + z = 0 \end{cases} \implies \begin{cases} 3z + 3w = 0 \\ y - 2z = 0 \\ x + w = 0 \\ 3z = 0 \end{cases}$$

To obtain the second system from the first system, we subtract the third equation from the first one and subtract the second equation from the fourth equation. The last equation of the second system implies that $z = 0$*. Now from the first equation we conclude that* $w = 0$*. Consequently, we can obtain from the second and the third equations that* $y = 0$ *and* $x = 0$*. Therefore the given four vectors are linearly independent and* S *is a basis.*

Grading: (10 points) *If you just show that the set* S *is linearly independent (or generating) without explaining why this implies that* S *is a basis, you lose* 3 *points. You might lose up to* 2 *points, if it is not clear from your solution that how the system is simplified.*

(iii) $n = 3$ *and* $S = \left\{ \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 9 \end{bmatrix} \right\}.$

Solution. *The dimension of* $\mathbb{R}^3$ *is equal to* 3*. Thus* S *cannot be a basis for* $\mathbb{R}^3$ *because it has four elements.*

Grading: (5 points) *To get the complete score you need to specify that* $\dim(\mathbb{R}^3) = 3$*. Otherwise you lose* 2 *points. Also, you would lose* 1 *point if you say* $\dim(S) = 4$ *instead of saying that* S *has* 4 *elements.*

2. *Suppose $T : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation given as $T(x) = Ax$ where A is the matrix*

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 1 & -3 & -4 & -1 \\ 1 & 0 & -1 & -1 \end{bmatrix}$$

(i) *Determine all vectors in* $\ker(T)$.

Solution. *Suppose* $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ *is in the kernel of T. Then we have:*

$$\begin{cases} x + 2y + z - w = 0 \\ x - 3y - 4z - w = 0 \\ x - z - w = 0 \end{cases} \implies \begin{cases} x + 2y + z - w = 0 \\ -5y - 5z = 0 \\ -2y - 2z = 0 \end{cases} \implies \begin{cases} x + 2y + z - w = 0 \\ y + z = 0 \end{cases}$$

$$\implies \begin{cases} x - z - w = 0 \\ y + z = 0 \end{cases}$$

The second system is obtained from the first by subtracting the first equation from the second and the third equations. Then rescaling the second and the third equations in the second system gives the second equation in the third system. Finally, we subtract twice the second equation from the first first equation in the third system to obtain the fourth system. Now if we pick z and w arbitrarily, then the first equation can be used to determine x and the second equation can be used to determine y. Therefore, a general solution to the above system is given as:

$$\ker(T) = \left\{ \begin{bmatrix} s+t \\ -s \\ s \\ t \end{bmatrix} \mid s,t \in \mathbb{R} \right\}. \tag{1}$$

Grading: (15 points) *Setting up the system for characterizing the elements in the kernel has* 5 *points. The remaining* 10 *points are for solving the system.*

(ii) *Compute* $\text{Nullity}(T)$.

Solution. *Part (i) shows that an arbitrary element in the kernel can be written as*

$$s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So the vectors $B = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *give a generating set for the kernel. These vectors are linearly independent because*

$$s \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0 \implies \begin{cases} s + t = 0 \\ -s = 0 \\ s = 0 \\ t = 0 \end{cases} \implies s, t = 0$$

So B is a basis for $\ker(T)$, *and hence* $\text{Nullity}(T) = 2$.

Grading: (10 points) *To get the complete score, you have to give an argument that the dimension of the kernel is equal to* 2 *by giving a basis for the kernel. If you give the final answer of part (i) in the form* (1) *and say that* $\text{Nullity}(T) = 2$ *without giving a basis you get* 5 *points. If you give a spanning set for the kernel, but don't specify that they are linearly independent, then you lose* 1 *point.*

(iii) *Compute* $\text{Rank}(T)$.

Solution. *The dimension formula implies that*

$$\text{Rank}(T) + \text{Nullity}(T) = \dim(\mathbb{R}^4) \implies \text{Rank}(T) + 2 = 4 \implies \text{Rank}(T) = 2.$$

Grading: (5 points)

3. *Recall that $M_{2\times 2}(\mathbb{R})$ denotes the vector space of 2×2 matrices. A standard basis for $M_{2\times 2}(\mathbb{R})$ is given as*

$$S = \left\{ f_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, f_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, f_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

(You don't need to show that S is a basis.) Suppose $R : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is a map defined as

$$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(i) *Show that R is a linear map.*

Solution. *Suppose $M_1, M_2 \in M_{2\times 2}(\mathbb{R})$ and $c \in \mathbb{R}$. Then we have*

$$R(cM_1 + M_2) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot (cM_1 + M_2) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot (cM_1) + \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot M_2 =$$

$$c\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot M_1 + \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot M_2 = cR(M_1) + R(M_2).$$

Therefore, R is a linear map.

Grading: (10 points)

(ii) *Write down the matrices $R(f_i)$ as a linear combination of the matrices in $S = \{f_1, f_2, f_3, f_4\}$.*

Solution. *Suppose $M_1, M_2 \in M_{2\times 2}(\mathbb{R})$ and $c \in \mathbb{R}$. Then we have*

$$R(f_1) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 2f_1$$

$$R(f_2) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2f_2$$

$$R(f_3) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = -f_1$$

$$R(f_4) = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = -f_2$$

Grading: (10 points) *If you just write down $R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$ (rather than $R(f_i)$) in terms of basis elements, you lose 5 points.*

(iii) *Find the 4×4 matrix $[R]_S^S$, which is the matrix representation of R with respect to the basis S. (You should be able to find $[R]_S^S$ using your answer to part (ii).)*

Solution. *We have*

$$[R]_S^S \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Grading: (10 points) *You get 6 points out of 10, if in your answer you mix up the rows and columns of $[R]_S^S$, i.e., giving the transpose of the above matrix as the final answer.*

4. *(i) Give a* 1*-dimensional subspace of*

$$V = \left\{A \in M_{2\times 2}(\mathbb{R}) \mid A^t = A\right\}$$

Here A^t denotes the transpose of A. So V is the space of symmetric 2×2 matrices. You may give your answer as multiples of a vector in V. Justify why it is a 1-dimensional subspace.

Solution. *Consider the space of matrices of the following form*

$$W = \left\{ \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} \mid t \in \mathbb{R} \right\} = \mathrm{Span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a non-zero symmetric matrix, the vector space generated by this matrix is a 1-dimensional subspace of V.

Grading: (10 points) *Giving a non-zero element of V worths* 5 *points. Giving a correct description of the subspace W given by the element worths* 5 *other points.*

(ii) Give a 2*-dimensional subspace of*

$$C(\mathbb{R},\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid \ f \text{ is a continuous function}\}.$$

You may give your answer as linear combinations of two elements of $C(\mathbb{R},\mathbb{R})$. Justify why it is a 2-dimensional subspace.

Solution. *Consider the space of functions of the following form*

$$W = \{s + tx \mid s,t \in \mathbb{R}\} = \mathrm{Span}(1,x)$$

where 1 *is the constant function* 1 *and x is the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$. Since these two functions are non-zero and neither of them is a multiple of the other one, W is a 2-dimensional subspace of V.*

Grading: (10 points) *Giving two linearly independent elements of $C(\mathbb{R},\mathbb{R})$ worths* 6 *points. (In particular, if you give pairs of functions like $\{x,y\}$ or $\{x^2,y^2\}$ (in which the second one doesn't give an element of $C(\mathbb{R},\mathbb{R})$) you get only* 3 *points.) Giving a correct description of the subspace W given by these elements and justifying that it has the required properties worths* 4 *points.*