Take-Home Exam II

1. Suppose the matrix A is given as

$$A = \begin{bmatrix} 1 & 0 & 2 & 2 & 5 \\ -4 & -3 & -14 & 1 & 1 \\ -6 & 2 & -6 & 4 & 0 \\ 0 & 1 & 3 & 3 & 5 \end{bmatrix}.$$

   (i) Determine the rank of A.

   (ii) Find a basis for the kernel of the linear transformation $T_A : \mathbb{R}^5 \to \mathbb{R}^4$ given by $T_A(\vec{x}) = A\vec{x}$.

   (iii) Describe the image of T_A as the set of solutions to a system of equations.

2. (i) Determine all values of s,t such that the following system has a unique solution:

$$\begin{cases} 3x + 4y = t \\ sx + y = -7 \end{cases}$$

(ii) Determine all values of s,t such that the above system has no solution.

3. Consider the matrix Q given as

$$Q = \begin{bmatrix} 1 & 1 & 2 & 8 & 1 \\ -2 & -1 & -1 & -16 & -1 \\ 1 & 0 & -2 & 14 & 0 \\ 1 & 1 & 2 & 9 & 8 \\ 1 & 1 & 2 & 8 & 2 \end{bmatrix}.$$

(i) By computing the determinant of Q shows that the columns of Q form a basis $\mathcal{S}$ of $\mathbb{R}^5$. Give the steps of computing the determinant.

(ii) Find the inverse matrix Q^{-1}.

(iii) Let $T_A : \mathbb{R}^5 \to \mathbb{R}^5$ be the linear transformation $T_A(\vec{x}) = A\vec{x}$ where A is the matrix

$$A = \begin{bmatrix} -6 & -1 & 2 & 0 & 1 \\ 2 & 0 & -1 & -2 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ -2 & 1 & 0 & -3 & 1 \\ 1 & 2 & 2 & 0 & -5 \end{bmatrix}.$$

Find the matrix representation $[T_A]_{\mathcal{S}}$ of T_A.

4. The following information about a matrix B is given.

- The reduced row echelon form of B is equal to
$$\begin{bmatrix} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & 8 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- The matrix B maps the vector $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 14 \\ 4 \\ 7 \end{bmatrix}$;

- The matrix B maps the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$;

- The matrix B maps the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}$.

(i) Determine the row space of B.

(ii) Determine the column space of B.

5. Suppose $\mathcal{P}_n$ denotes the space of all polynomials of degree at most n. That is to say, any element of $\mathcal{P}_n$ can be written as:

$$a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Let $T : \mathcal{P}_4 \to \mathcal{P}_4$ be the map given by

$$T(p)(x) := xp'(x) + p'(x)$$

where $p(x) \in \mathcal{P}_n$. Find all eigenvalues of T. Then show that T is diagonalizable by finding a basis for $\mathcal{P}_4$ which only consists of eigenvectors.

6. For each real number t, determine whether the following matrix is diagonalizable. Justify why you think the matrix is diagonalizable or non-diagonalizable. In the case that it is diagonalizable find the diagonal form.

$$\begin{bmatrix} t & 0 & 0 & 7 & -3 \\ 1 & 2 & 0 & 2 & 4 \\ 1 & 1 & 3 & -6 & 8 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

7. (a) Verify the following identity:

$$\det \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & a_4 \end{bmatrix} = (a_1 - b_1)(a_2 - b_2)(a_3 - b_3)a_4$$

(Hint: Using elementary row operations turn your matrix into a lower or upper triangular matrix.)

(b) Generalize the last part by finding a similar formula for the determinant of the following $n \times n$ matrix and justify your answer.

$$\begin{bmatrix} a_1 & a_2 & \dots & a_{n-2} & a_{n-1} & a_n \\ b_1 & a_2 & \dots & a_{n-2} & a_{n-1} & a_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \dots & b_{n-2} & a_{n-1} & a_n \\ b_1 & b_2 & \dots & b_{n-2} & b_{n-1} & a_n \end{bmatrix}$$