Take-Home Exam I

1. Suppose $M \in M_{2\times 2}(\mathbb{R})$ is an arbitrary 2×2 matrix and $T_M : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is a map defined as

$$T_M\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = M \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot M$$

   (i) Show that T_M is a linear map.

   (ii) Show that T_M is not surjective for any choice of M.

   (iii) Find an example of M such that the rank of T_M is equal to 2.

2. (i) Suppose $\mathcal{D}_n$ denotes the following sets of matrices in $M_{n\times n}(\mathbb{R})$:

$$\mathcal{D}_n = \left\{ A = [a_{i,j}] \in M_{n\times n}(\mathbb{R}) \mid \; a_{i,j} = 0 \text{ if } i \neq j \text{ and } \sum_{i=1}^{n} a_{i,i} = 0 \right\}.$$

So $\mathcal{D}_n$ is the set of diagonal matrices such that the sum of diagonal terms is equal to 0. Show that $\mathcal{D}_n$ is a subspace of $M_{n\times n}(\mathbb{R})$. Find a basis for $\mathcal{D}_n$ and determine its dimension.

(ii) Suppose $\mathcal{S}_n$ denotes the following sets of matrices in $M_{n\times n}(\mathbb{R})$:

$$\mathcal{S}_n = \left\{ A = [a_{i,j}] \in M_{n\times n}(\mathbb{R}) \mid A = A^t, \sum_{i=1}^{n} a_{i,i} = 0 \right\}.$$

So $\mathcal{S}_n$ is the set of symmetric matrices such that the sum of diagonal terms is equal to 0. Show that $\mathcal{S}_n$ is a subspace of $M_{n\times n}(\mathbb{R})$. Find a basis for $\mathcal{S}_n$ and determine its dimension. (Hint: If you find it hard to answer the question for arbitrary n, firstly look at the problem for small values of n, and after finding a pattern consider the general case.)

3. Suppose $\mathcal{P}_n$ denotes the space of all polynomials of degree at most n. That is to say, any element of $\mathcal{P}_n$ can be written as:

$$a_0 + a_1 x + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Then $\mathcal{P}_n$ is a vector space with basis $S_n = \{1, x, \ldots, x^n\}$. (You don't need to prove this.) Let $T : \mathcal{P}_n \to \mathcal{P}_{n+1}$ be the map given by

$$T(p)(x) := (2x+1) \cdot p(x) + xp'(x) - \int_0^x p(s)ds$$

where $p(x) \in \mathcal{P}_n$. Find the matrix representation $[T]_{S_n}^{S_{n+1}}$ of the linear transformation T. (Hint: As in the last problem, if you find it hard to answer the question for arbitrary n, firstly look at the problem for small values of n, and after finding a pattern consider the general case.)