Homework IX

1. Find the Taylor series of the function $f(x) = \cos(x)$, centered at the point $a = \frac{\pi}{3}$. Show that this Taylor series is equal to $\cos(x)$ for all values of x.

2. Use power series representations of functions to evaluate the following limits:

   (a) $\lim_{x \to 0} \dfrac{x - \sin(x)}{x^3}$.

   (b) $\lim_{x \to 0} \dfrac{1 - \cos(x)}{1 + x - e^x}$

3. (a) Find a power series representation for the function $f(x) = x\sin(x^5)$ centered at 0.

(b) Use part (a) to evaluate $f^{50}(0)$.

(c) Use part (a) to evaluate $f^{36}(0)$.

4. (a) Find a power series representation for the function $f(x) = e^{-x^2}$.

(b) Find a power series representation for the function $g(x) = \int_0^x e^{-t^2}\,dt$. Where is this representation valid?

(c) Use the last part to write a convergent series whose sum is equal to $\int_0^{10} e^{-t^2}\,dt$.

(d) We use the sum of the first 5 terms of the series of the previous part to approximate the integral $\int_0^{10} e^{-t^2}\,dt$. Can you give an upper bound for the error of this approximation? (Hint: Use alternating series estimation Theorem.)

5. Use power series representations for functions that we studied in the class to evaluate the following sums:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n! \cdot 3^n}$

(b) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n \cdot 3^n}$

(c) $\sum_{n=1}^{\infty}\frac{n}{2^n}$ (Hint: By taking the derivative of the power series representation of the function $\frac{1}{1-x}$ find a function which is equal to $\sum_{n=1}^{\infty} nx^n$.)

(d) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ (Hint: Starting with the last part, find a function which is equal to $\sum_{n=1}^{\infty} n^2 x^n$.)