1. Compute the following integrals:

(a) $\int \frac{10x-19}{x^2-x-12}\,dx$

$x^2-x-12 = (x-4)(x+3)$

$\frac{10x-19}{x^2-x-12} = \frac{A}{x-4} + \frac{B}{x+3} = \frac{A(x+3)+B(x-4)}{x^2-x-12}$

$\Rightarrow 10x-19 = A(x+3)+B(x-4)$

$x: -3 \Rightarrow -49 = -7 \cdot B \Rightarrow B = 7$

$x: 4 \Rightarrow 21 = 7 \cdot A \Rightarrow A = 3$

$\Rightarrow \int \frac{10x-19}{x^2-x-12}\,dx = \int \frac{3}{x-4} + \frac{7}{x+3}\,dx = 3\ln(|x-4|) + 7\ln(|x+3|) + C$

(b) $\int_1^e \frac{(\ln(x))^2+1}{x}\,dx$

$u = \ln(x)$

$du = \frac{1}{x}$

$= \int_{\ln(1)}^{\ln(e)} u^2+1\,du$

$= \frac{u^3}{3} + u \Big|_0^1 = \frac{1}{3} + 1 = \frac{4}{3}$

2. A large tank contains 50 kg of salt dissolved in 10000 liters of water. A solution of water and salt containing 0.02 kg of salt per liter enters the tank through a pipe at a rate of 3 liters per minute. At the same time, the mixture of salt and water is pumped out of the tank at a rate of 3 liters per minute. The tank is kept well mixed and the concentration of salt in the tank is uniform. Let $S(t)$ denote the amount of salt in the tank after t minutes, measured in kg .

(a) Write down a differential equation for $S(t)$.

rate of change of S = rate in of S − rate out of S

rate in of S = (rate in of Solution) × (Concentration of Salt) = 3 lit/min × 0.02 kg/lit = 0.06 kg/min

rate out of S = (rate out of Solution) × (Concentration of Salt) = 3 lit/min × $\frac{S(t)}{10000}$ = 0.0003 S

$$\Rightarrow \boxed{\frac{dS}{dt} = 0.06 - 0.0003S}$$

(b) Write down an initial condition for $S(t)$.

$$S(0) = 50 \text{ kg}$$

(c) Solve the differential equation with the initial condition given in parts (a) and (b).

$$\frac{dS}{dt} + 0.0003\,S = 0.06$$

$$P(t) = 0.0003 \Rightarrow I(t) = e^{\int 0.0003\,dt} = e^{0.0003t}$$

(multiply by $e^{0.0003t}$)

$$\Rightarrow \left(e^{0.0003t}\frac{dS}{dt} + 0.0003\,e^{0.0003t}S\right) = 0.06\,e^{0.0003t}$$

$$\Rightarrow \frac{d}{dt}\left(e^{0.0003t}\cdot S\right) = 0.06\,e^{0.0003t}$$

$$\Rightarrow e^{0.0003t}\cdot S = \int 0.06\,e^{0.0003t}\,dt = \frac{0.06}{0.0003}e^{0.0003t} + C$$

$$\Rightarrow e^{0.0003t}\cdot S = 200\,e^{0.0003t} + C \Rightarrow S(t) = 200 + Ce^{-0.0003t}$$

initial condition $\Rightarrow 50 = 200 + C \Rightarrow \boxed{C = -150} \Rightarrow \boxed{S(t) = 200 - 150\,e^{-0.0003t}}$

Another way to solve this differential equatio is to identify it as a Separable equation.

3. (a) Find the interval of convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{5^n (x+1)^n}{\sqrt{n+4}}$$

Center $= -1$

radius of convergence:

$$\lim_{n\to\infty} \left| \frac{\frac{5^{n+1}(x+1)^{n+1}}{\sqrt{n+5}}}{\frac{5^n (x+1)^n}{\sqrt{n+4}}} \right| = \lim_{n\to\infty} |5(x+1)| \sqrt{\frac{n+4}{n+5}}$$

$$= \cancel{\lim} 5|x+1| \sqrt{\lim_{n\to\infty} \frac{n+4}{n+5}} \xrightarrow[\text{rule}]{\text{L'Hospital's}} 5|x+1|$$

ratio test: ~~Power series is convergent~~ $5|x+1|<1 \Rightarrow$ Power series is convergent

$5|x+1|>1 \Rightarrow$ Power series is divergent

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(b) Find the Maclaurin series of $f(x) = xe^{\frac{1}{2}x^4}$. What is the radius of convergence?

function	Power series representation
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
$e^{\frac{1}{2}x^4}$	$1 + \frac{1}{2}x^4 + \frac{x^8}{2^2 \times 2!} + \frac{x^{12}}{2^3\, 3!} + \frac{x^{16}}{2^4\, 4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{4n}}{2^n \times n!}$
$xe^{\frac{1}{2}x^4}$	$x + \frac{1}{2}x^5 + \frac{x^9}{2^2 \times 2!} + \frac{x^{13}}{2^3 \times 3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{4n+1}}{2^n \times n!}$

Since a power series representation is the Taylor series, Taylor series of $xe^{\frac{1}{2}x^4}$ is equal to $\sum_{n=0}^{\infty} \frac{x^{4n+1}}{2^n \times n!}$.

This series is convergent for all values of x, because the power series for e^x is convergent for all values of x.

$5|x+1|=1$: ratio test is inconclusive

$$5|x+1|=1 \Longleftrightarrow 5(x+1)=\pm 1 \Rightarrow x=-\frac{6}{5} \text{ or } -\frac{4}{5}$$

we have to check these points separately:

$x=-\frac{6}{5}$:

$$\sum_{n=1}^{\infty} \frac{5^n\left(1+\frac{-6}{5}\right)^n}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{\cancel{5^n} \times \frac{(-1)^n}{\cancel{5^n}}}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+4}}$$

This series is ~~convergent~~ alternating. Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+4}}=0 \quad \& \quad \frac{1}{\sqrt{n+5}} < \frac{1}{\sqrt{n+4}}$$

Thus the series is convergent by the alternating Series test

$x=\frac{-4}{5}$:

$$\sum_{n=1}^{\infty} \frac{5^n\left(1+\frac{-4}{5}\right)^n}{\sqrt{n+4}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

Consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent because it's a p-series with $p=\frac{1}{2}$.

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+4}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+4}} = \sqrt{\lim_{n\to\infty} \frac{n}{n+4}} \xlongequal{\text{L'Hospital's rule}} 1$$

limit compariso test $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ is divergent $\Rightarrow$ Interval of convergence $\left[\frac{-6}{5}, -\frac{4}{5}\right)$

4. (I) Which one of the following options gives a convergent improper integral?

(a) $\int_0^1 \frac{1}{x+1}\,dx$

(b) $\int_0^1 \frac{1}{x^2}\,dx$

(c) $\int_1^{\infty} \frac{1}{x^2+1}\,dx$ $= \lim_{t\to\infty} \arctan(x)\Big|_1^t = \lim_{t\to\infty} \arctan(t) - \pi/4 = \frac{\pi}{2} - \pi/4 = \frac{\pi}{4}$

(d) $\int_1^{\infty} \frac{1}{x+1}\,dx$

(II) We are looking for the Maclaurin series of a function $f(x)$. We know that $f^{(20)}(0) = 19! \cdot 2^{19}$. What is the coefficient of x^{20} in the Maclaurin series of $f(x)$?

(a) $20! \cdot 19! \cdot 2^{19}$ (b) $\frac{1}{20 \cdot 2}$ (c) $\frac{2^{19}}{20}$ (d) $19! \cdot 2^{19}$

Recall that the coefficient of x^n in a power series $\sum_{i=0}^{\infty} a_i x^i$ defined to be a_n.

Coefficient of x^{20} in Maclaurin series $=$ $19! \cdot 2^{19}/20! = \frac{2^{19}}{20}$

(III) A power series is convergent at -1 and divergent at 1. Which one of the following points could be in the interval of the convergence of this power series:

(a) 2 (b) -2 (c) 4 (d) 10

Since the convergence points is always an interval, the only possibility is -2.

(IV) Write down a convergent **geometric** series with infinitely many non-zero terms such that the sum is equal to e.

$$\frac{e}{2} + \frac{e}{4} + \frac{e}{8} + \cdots = \sum_{n=1}^{\infty} \frac{e}{2^n} = e$$

5. We want to use a Taylor polynomial of $f(x) = \sqrt[3]{x}$ to approximate $\sqrt[3]{25}$.

(a) What would be a good choice for the center of the Taylor polynomial? Why?

27, because we can easily evaluat $f(x)$ (and its derivatives) at $x=27$. Moreove, this point is close to 25.

(b) Use the Taylor polynomial of degree 2 centered at the point that you found in the previous part to write an estimate for $\sqrt[3]{25}$.

$$f(x) = \sqrt[3]{x}$$

$$f'(x) = \frac{1}{3} x^{-2/3}$$

$$f''(x) = \frac{1}{3} \times \left(-\frac{2}{3}\right) x^{-5/3}$$

$$f'''(x) = \frac{1}{3} \times \left(-\frac{2}{3}\right) \times \left(-\frac{5}{3}\right) x^{-8/3}$$

$$\Rightarrow f(27) = 3, \quad f'(27) = \frac{1}{27}, \quad f''(27) = \frac{-2}{3^7}$$

$$\Rightarrow T_2(X) = f(27) + f'(27)(x-27) + \frac{f''(27)}{2!}(x-27)^2$$

$$\Rightarrow T_2(X) = 3 + \frac{1}{27}(x-27) - \frac{2}{3^7 \times 2!}(x-27)^2$$

$$\Rightarrow \boxed{f(25) \approx T_2(25) = 3 - \frac{2}{27} - \frac{4}{3^7}}$$

(c) Find an upper bound for the error of your estimate.

Taylor's inequality:

$[27-2, 27+2]$

$$|f(x) - T_2(x)| \leq M \frac{|x-27|^3}{3!} \quad \text{for } x \text{ in } = [25, 29]$$

where is an upper bound for $|f'''(x)| = \frac{10}{27} |x^{-8/3}|$ on the interval $[25, 29]$. $\frac{10}{27} x^{-8/3}$ is a decreasing positive function on this interval

So we can pick $M = |f'''(25)| = \frac{10}{27} 25^{-8/3}$

$$\Rightarrow \underbrace{|f(25) - T_2(25)|}_{\text{error of the approximation}} \leq \frac{10}{27} 25^{-8/3} \cdot \frac{2^3}{3!}$$

6. (a) Solve the following initial value problem:

$$xy' + y^2 = 2x^2y^2 \qquad y(1) = 0$$

$xy' = (2x^2-1)y^2 \Rightarrow \frac{dy}{dx} = \left(\frac{2x^2-1}{x}\right)y^2$

we can divide by y^2 if $y \neq 0$:

$\Rightarrow \frac{dy}{y^2} = (2x - \frac{1}{x})\,dx \Rightarrow \int \frac{dy}{y^2} = \int 2x - \frac{1}{x}\,dx \Rightarrow -\frac{1}{y} = x^2 - \ln(|x|) + C$

$\Rightarrow y = \frac{-1}{x^2 - \ln(|x|) + C}$

If $y = 0$: $x\frac{dy}{dx} + y^2 = 2x^2y^2$. So $y = 0$ is a solution, too. The general solution: $y = \frac{-1}{x^2 - \ln(|x|) + C}$ or $y = 0$

$y(x) = 0$ is the only one that satisfies $y(1) = 0$

(b) Find the general solution of the following differential equation:

$$(x^2 + 1)y' + 3x^3y = 6xe^{-\frac{3}{2}x^2}$$

$y' + \frac{3x^3}{x^2+1}y = \frac{6x}{x^2+1}e^{-3/2\,x^2}$

$P(x) = \frac{3x^3}{x^2+1} \qquad Q(x) = \frac{6x}{x^2+1}e^{-\frac{3}{2}x^2}$

$\int p(x)\,dx = \int \frac{3x^3}{x^2+1}dx = \frac{3}{2}\int \frac{(u-1)}{u}du \qquad u = x^2+1, \; du = 2x\,dx$

$= \frac{3}{2}\int 1 - \frac{1}{u}\,du = \frac{3}{2}(u - \ln(|u|)) + C =$

$= \frac{3}{2}(x^2 - \ln(x^2+1)) + \underbrace{C + \frac{3}{2}}_{C'}$

we pick c' to be 0. Therefore, our integrating factor is:

$I(x) = e^{\int p(x)dx} = e^{\frac{3}{2}x^2} \times e^{-\frac{3}{2}\ln(x^2+1)} = \frac{e^{3/2\,x^2}}{(x^2+1)^{3/2}}$

We multiply our differential equation by the integrating factor $I(t)$:

$$\frac{e^{3/2x^2}}{(x^2+1)^{3/2}}\frac{dy}{dx} + \frac{3e^{3/2x^2}x^3}{(x^2+1)^{5/2}}y = \frac{6x}{(x^2+1)^{5/2}}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{e^{3/2x^2}}{(x^2+1)^{3/2}}y\right) = \frac{6x}{(x^2+1)^{5/2}}$$

$u = x^2+1$

$du = 2x\,dx$

$$\Rightarrow \frac{e^{3/2x^2}}{(x^2+1)^{3/2}}\cdot y = \int \frac{6x}{(x^2+1)^{5/2}}dx = \int \frac{3\,du}{u^{5/2}}$$

$$= \cancel{3}\left(\frac{-2}{\cancel{3}}\right)u^{-3/2} + C$$

$$= -2(x^2+1)^{-3/2} + C$$

$$\Rightarrow y = \frac{(x^2+1)^{3/2}}{e^{3/2x^2}}\left(-2(x^2+1)^{-3/2} + C\right)$$

$$\Rightarrow \boxed{y(x) = -2e^{-3/2x^2} + C(x^2+1)^{3/2}e^{-3/2x^2}}$$

7. (a) Determine whether the following series is convergent or divergent. If it converges, find its sum.

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!3^{2n}}$$

The power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ is convergent for all values of x and is equal to $\cos(x)$. In particular, it is convergent at $x=\frac{1}{3}$ and is equal to $\cos(\frac{1}{3})$.

The evaluation of $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ at $\frac{1}{3}$ gives the above series.

(b) Determine whether the following series is convergent or divergent. State the test you use, and show your work:

$$\sum_{n=2}^{\infty}\frac{2+\cos(n^2)}{\sqrt{n}-1}$$

$$2+\cos(n^2) \geq 2-1=1$$

$$\Rightarrow \frac{2+\cos(n^2)}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}} \geq 0$$

The series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ is divergent because it is a p-series with $p=\frac{1}{2}$. Therefore, $\sum_{n=2}^{\infty} \frac{2+\cos(n^2)}{\sqrt{n}-1}$

is also divergent.

(c) We wish to approximate the sum of the following series with error less than 0.0001.

$$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

Determine a partial sum of this series which gives such an approximation. Justify your answer.

If we use S_i (the i^{th} partial sum) then the error by the integral test is less than:

$$\text{error} \leq \int_i^{\infty} \frac{1}{x^5} dx = \lim_{t \to \infty} \frac{-x^{-4}}{4} \Big|_i^t$$

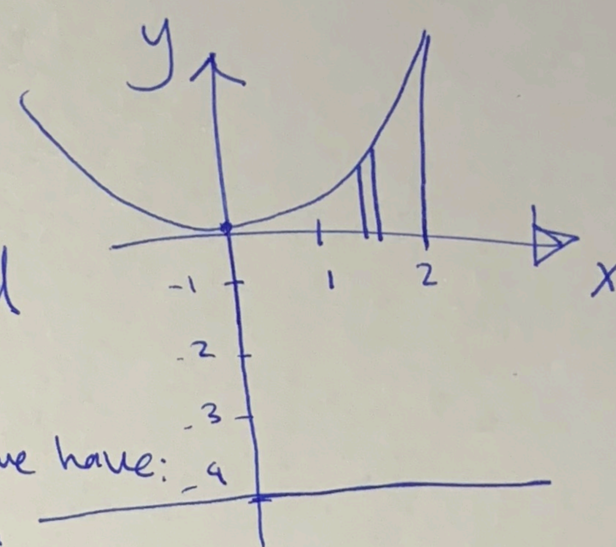
$$= \frac{i^{-4}}{4} = \frac{1}{4i^4}$$

So if $\frac{1}{4 \cdot i^4} < \frac{1}{10000}$, we can ensure that the error is less than 0.0001.

This inequality $\left(\frac{1}{4 \cdot i^4} < \frac{1}{10000}\right)$ holds for $i = 8$. Therefore, we can use S_8 to obtain such an approximation.

(In fact, we obtain a good approximation using S_i for any $i \geq 8$.)

8. (a) Let $\mathcal{R}$ be the region enclosed by the x-axis, $x = 2$ and $y = x^2$. Find the volume obtained by rotating $\mathcal{R}$ about the line $y = -4$.



we slice the region vertically. A typical slice is a washer with inner radius 4 and outer radius $4+x^2$. Our variable is x which varies from 0 to 2. Therefore, we have:

$$\text{volume} = \int_0^2 \pi\left[(4+x^2)^2 - 4^2\right] dx = \pi\int_0^2 x^4 + 8x^2\, dx$$

$$= \pi\left[\frac{x^5}{5} + \frac{8x^3}{3}\Big|_0^2\right] = \pi\left(\frac{32}{5} + \frac{64}{3}\right) = \boxed{\pi\,\frac{32\times 13}{15}}$$

(b) For $\frac{\pi}{4} \le t \le \frac{\pi}{3}$, let $g(t) = \int_{\frac{\pi}{2}}^{t} \sqrt{\tan^2(x) - 1}\, dx$. Find the length of the graph of this function for $\frac{\pi}{4} \le t \le \frac{\pi}{3}$.

$$g'(t) = \sqrt{\tan^2(t) - 1}$$ Fundamental theorem of calculus

$$\sqrt{1 + g'(t)^2} = \sqrt{1 + (\tan^2(t) - 1)} = \tan(t) \quad \text{for } \pi/4 \le \pi/3$$

$$\Rightarrow \text{arc length} = \int_{\pi/4}^{\pi/3} \tan(t)\, dt = \int_{\pi/4}^{\pi/3} \frac{\sin(t)}{\cos(t)}\, dt$$

$$= \int_{\cos(\pi/4)}^{\cos(\pi/3)} \frac{-du}{u} = -\ln(|u|)\Big|_{\cos(\pi/4)}^{\cos(\pi/3)} \qquad u = \cos(t), \quad du = -\sin(t)\,dt$$

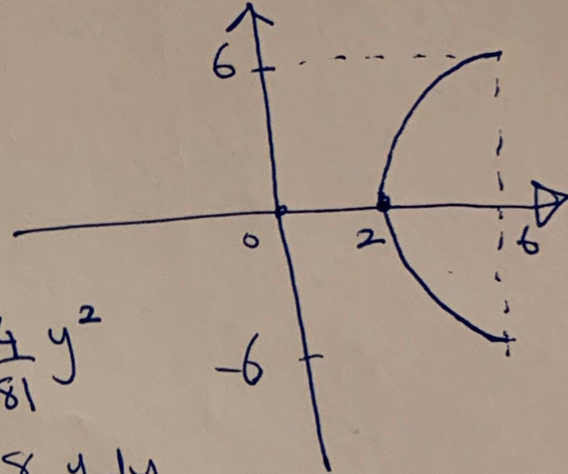
$$= \ln(\cos(\pi/4)) - \ln(\cos(\pi/3)) = \ln\left(\frac{\sqrt{2}}{2}\right) - \ln\left(\frac{1}{2}\right) = \ln(\sqrt{2})$$

(c) Find the area of the surface obtained by rotating the curve $9x = y^2 + 18$, $2 \leq x \leq 6$, about the x axis.

$$x = \frac{y^2+18}{9} = f(y) \qquad f'(y) = \frac{2}{9}y$$

$$\text{area} = \int_0^6 2\pi y \sqrt{1 + f'(y)^2}\, dy$$

$$= 2\pi \int_0^6 y \sqrt{1 + \frac{4}{81}y^2}\, dy$$



$$= 2\pi \int_1^{1+\frac{4}{81}\cdot 6^2} \frac{81}{8} \sqrt{u}\, du \qquad u = 1 + \frac{4}{81}y^2$$

$$du = \frac{8}{81} y\, dy$$

$$= \frac{81\pi}{4} \int_1^{1+\frac{16}{9}} \sqrt{u}\, du$$

$$= \frac{81\pi}{4}\left(\frac{2}{3} u^{3/2} \Big|_1^{\frac{25}{9}}\right) = \frac{27\pi}{2}\left(\left(\frac{25}{9}\right)^{3/2} - 1\right)$$

$$= \frac{27\pi}{2}\left[\frac{125}{27} - 1\right]$$

$$= \boxed{49\pi}$$