Calculus II

Midterm 1 Fall 2018

Name: ______________________ **ID:** ______________________

Instructions:

(1) Fill in your name and Columbia University ID at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. 1/2, not 2/4) to receive full credit.

(4) Answer all questions in the space provided. If you need more room use the blank backs of the pages.

(5) Show your work; correct answers alone will receive only partial credit.

(6) This exam has 5 extra credit points.

Problem	1 (10 pts)	2 (10 pts)	3 (10 pts)	4 (10 pts)	5 (10 pts)	6 (10 pts)	7 (15 pts)	8 (25 pts)	Total (100 pts)
Score									

Evaluate the following integrals. Each part worths 10 points:

1. $\displaystyle\int_0^1 \frac{\arctan^2(x)+1}{x^2+1}\,dx$

   Solution. We use u-substitution where $u = \arctan(x)$. Then $du = \frac{1}{x^2+1}dx$ and we have:

$$
\begin{aligned}
\int_0^1 \frac{\arctan^2(x)+1}{x^2+1}\,dx &= \int_{\arctan(0)}^{\arctan(1)} u^2+1\,du \\
&= \int_0^{\frac{\pi}{4}} u^2+1\,du \\
&= \frac{u^3}{3}+u\Big|_0^{\frac{\pi}{4}} \\
&= \frac{(\frac{\pi}{4})^3}{3}+\frac{\pi}{4} \\
&= \frac{\pi^3}{192}+\frac{\pi}{4}
\end{aligned}
$$

2. $\displaystyle\int e^{3t}\cos(2t)\,dt$

Solution. We use integration by parts where $u = e^{3t}$ and $dv = \cos(2t)dt$. That implies that $du = 3e^{3t}dt$ and $v = \frac{\sin(2t)}{2}$. Therefore, we have:

$$\begin{aligned}\int e^{3t}\cos(2t)\,dt &= e^{3t}\frac{\sin(2t)}{2} - \int 3e^{3t}\frac{\sin(2t)}{2}\,dt \\ &= e^{3t}\frac{\sin(2t)}{2} - \frac{3}{2}\int e^{3t}\sin(2t)\,dt \qquad (1)\end{aligned}$$

Then we apply integration by parts again to the integral in (1) where $u = e^{3t}$ and $dv = \sin(2t)dt$. We have $du = 3e^{3t}dt$ and $v = -\frac{\cos(2t)}{2}$ which implies that:

$$\begin{aligned}e^{3t}\frac{\sin(2t)}{2} - \frac{3}{2}\int e^{3t}\sin(2t)\,dt &= e^{3t}\frac{\sin(2t)}{2} - \frac{3}{2}(-e^{3t}\frac{\cos(2t)}{2} - \int -3e^{3t}\frac{\cos(2t)}{2}\,dt) \\ &= e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) - \frac{9}{4}\int e^{3t}\cos(2t)\,dt)\end{aligned}$$

In summary, we have:

$$\int e^{3t}\cos(2t)\,dt = e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) - \frac{9}{4}\int e^{3t}\cos(2t)\,dt)$$

which implies that:

$$(1 + \frac{9}{4})\int e^{3t}\cos(2t)\,dt = e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) \implies$$

$$\frac{13}{4}\int e^{3t}\cos(2t)\,dt = e^{3t}\frac{\sin(2t)}{2} + \frac{3}{4}e^{3t}\cos(2t) \implies$$

$$\int e^{3t}\cos(2t)\,dt = \frac{2}{13}e^{3t}\sin(2t) + \frac{3}{13}e^{3t}\cos(2t) + C$$

At the end, we included the arbitrary constant of integration, because our integral is indefinite.

3. $\displaystyle\int_1^3 \frac{3x+1}{x^2-2x-15}\,dx$

Solution. The denominator of this fraction can be factorized as $(x-5)(x+3)$. Therefore, we can use partial fraction decomposition to compute this integral:

$$\frac{3x+1}{x^2-2x-15} = \frac{A}{x-5} + \frac{B}{x+3} \implies$$

$$\frac{3x+1}{x^2-2x-15} = \frac{A(x+3)+B(x-5)}{(x-5)(x+3)} \implies$$

$$3x+1 = A(x+3)+B(x-5) \tag{2}$$

Identity (2) has to hold for all values of x. In particular, we can evaluate it at $x=5$ and $x=-3$:

$$x=5: \qquad 3\times 5+1 = A(5+3)+B(5-5) \implies$$

$$16 = 8\times A \implies A = 2$$

$$x=-3: \qquad 3\times(-3)+1 = A(-3+3)+B(-3-5) \implies$$

$$-8 = -8\times B \implies B = 1$$

In order to find A and B, we can follow the following alternative approach. The equation (2) can be rewritten as:

$$3x+1 = (A+B)x + (3A-5B) \implies$$

$$\begin{cases} A+B=3 \\ 3A-5B=1 \end{cases}$$

By multiplying the first equation by 5 and then adding it up to the second equation, we obtain:

$$8A = 16 \implies A = 2.$$

Similarly, we can multiply the first equation by 3 and then subtract it from the second equation. This implies that:

$$-8B = -8 \implies B = 1.$$

In any case, we have:

$$\begin{aligned}
\int_1^3 \frac{3x+1}{x^2-2x-15}\,dx &= \int_1^3 \frac{2}{x-5} + \frac{1}{x+3}\,dx \\
&= 2\ln(|x-5|) + \ln(|x+3|)|_1^3 \\
&= (2\ln(|3-5|) + \ln(|3+3|)) - (2\ln(|1-5|) + \ln(|1+3|)) \\
&= 2\ln(2) + \ln(6) - 2\ln(4) - \ln(4) \\
&= \ln(\frac{2^2\times 6}{4^2\times 4}) \\
&= \ln(\frac{3}{8})
\end{aligned}$$

4. $\displaystyle\int \sin^2(x)\cos^4(x)\,dx$

Solution. This integral can be solved using trigonometric identities:

$$\begin{aligned}
\int \sin^2(x)\cos^4(x)\,dx &= \int (\sin(x)\cos(x))^2\cos^2(x)\,dx \\
&= \int \frac{\sin^2(2x)}{4}\frac{1+\cos(2x)}{2}\,dx \\
&= \int \frac{1-\cos(4x)}{8}\frac{1+\cos(2x)}{2}\,dx \\
&= \frac{1}{16}\int 1-\cos(4x)+\cos(2x)-\cos(4x)\cos(2x)\,dx \\
&= \frac{1}{16}\int 1-\cos(4x)+\cos(2x)-\frac{\cos(6x)+\cos(2x)}{2}\,dx \\
&= \frac{1}{32}\int 2-2\cos(4x)+\cos(2x)-\cos(6x)\,dx \\
&= \frac{1}{32}\left(2x-\frac{\sin(4x)}{2}+\frac{\sin(2x)}{2}-\frac{\sin(6x)}{6}\right)+C \\
&= \frac{x}{16}-\frac{\sin(4x)}{64}+\frac{\sin(2x)}{64}-\frac{\sin(6x)}{192}+C
\end{aligned}$$

5. $\displaystyle\int \frac{\cos(x)\sin(x)}{2-\cos(x)}\,dx$

Solution. We can use u-substitution with $u = \cos(x)$. Then $du = -\sin(x)dx$ and we have:

$$\int \frac{\cos(x)\sin(x)}{2-\cos(x)}\,dx = \int \frac{u}{u-2}\,du$$

The expression in inside the integral on the left hand side can be simplified as:

$$\frac{u}{u-2} = \frac{u-2+2}{u-2} = \frac{u-2}{u-2} + \frac{2}{u-2} = 1 + \frac{2}{u-2} \tag{3}$$

Therefore, we can write::

$$\begin{aligned}\int \frac{u}{u-2}\,du &= \int 1 + \frac{2}{u-2}\,du \\ &= \int 1\,du + 2\int \frac{1}{u-2}\,du \\ &= u + 2\ln(|u-2|) + C \\ &= \cos(x) + 2\ln(2-\cos(x)) + C\end{aligned}$$

In the last step, we plug in $\cos(x)$ for u.

6. $\displaystyle\int_0^2 t^3 e^{t^2}\, dt$

Solution. Firstly, use u-substitution with $u = t^2$. Then $du = 2tdt$ and we have:

$$\begin{aligned}\int_0^2 t^3 e^{t^2}\, dt &= \int_{0^2}^{2^2} ue^u \,\frac{du}{2} \\ &= \frac{1}{2}\int_0^4 ue^u \, du\end{aligned}$$

The latter integral can be computed using integration by parts. Let $r = u$ and $ds = e^u du$. Then $r = du$ and $s = e^u$, and we can rewrite the last expression as:

$$\begin{aligned}\frac{1}{2}\int_0^4 ue^u \, du &= \frac{1}{2}(ue^u|_0^4 - \int_0^4 e^u \, du) \\ &= \frac{1}{2}(4 \times e^4 - 0 \times e^0 - e^u|_0^4) \\ &= \frac{1}{2}(4e^4 - e^4 + e^0) \\ &= \frac{1}{2}(3e^4 + 1)\end{aligned}$$

7. (15 points) Albert's boomerang has the shape of the region enclosed by the parabolas $y = x^2 - 3x + 3$ and $y = 2x^2 - 6x + 5$. Find the area of his boomerang.

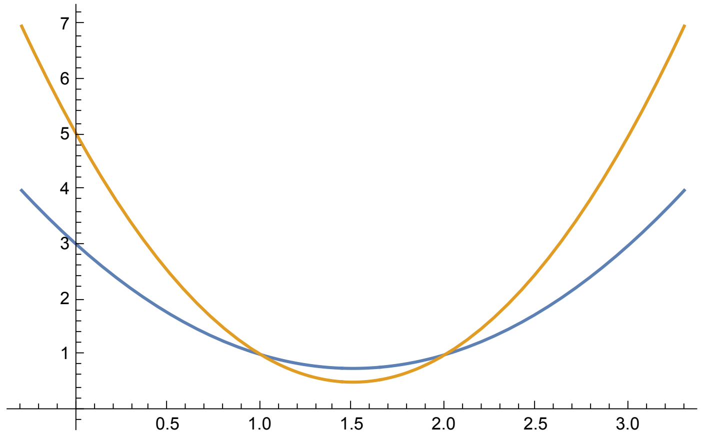
Solution. Firstly, we need to find the intersection points of the two parabolas. If (x,y) lies on the graph of these two curves, then:

$$x^2 - 3x + 3 = 2x^2 - 6x + 5 \implies$$

$$0 = x^2 - 3x + 2 \implies$$

$$x = 1, 2$$

Therefore, the two intersection points are (1,1) and (2,1). We slicing the region enclosed by the



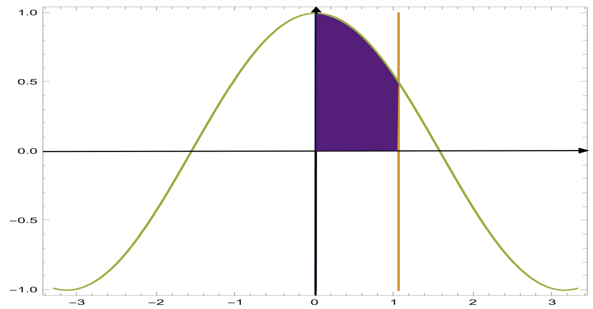
two parabolas vertically. Therefore, we have to use the x-coordinate to parametrize our slices and the possible values of x lie in the interval [1,2]. For $x \in [1,2]$, the length of the slice is equal to $(x^2 - 3x + 3) - (2x^2 - 6x + 5) = 3x - 2 - x^2$. (In order to see which graph is on top in the interval [1,2], we can evaluate our functions at an arbitrary point in (1,2) like $\frac{3}{2}$.) Therefore, the area is equal to:

$$\begin{aligned}\int_1^2 3x - 2 - x^2\, dx &= 3\frac{x^2}{2} - 2x - \frac{x^3}{3}\Big|_1^2 \\ &= (3 \times \frac{2^2}{2} - 2 \times 2 - \frac{2^3}{3}) - (3 \times \frac{1^2}{2} - 2 \times 1 - \frac{1^3}{3}) \\ &= (6 - 4 - \frac{8}{3}) - (\frac{3}{2} - 2 - \frac{1}{3}) \\ &= \frac{1}{6}\end{aligned}$$

8. (25 points) Let $\mathcal{R}$ be the region enclosed by the x-axis, y-axis, $x = \frac{\pi}{3}$, and the curve $y = \cos(x)$.

(a) Sketch the shape of this region in the coordinate plane.

Solution.



(b) Let $\mathcal{S}$ be the solid given by rotating the region $\mathcal{R}$ about the y-axis. Find the volume of $\mathcal{S}$.

Solution. We slice the region $\mathcal{R}$ vertically. Thus we have to use the x-axis to parametrize our slices, and for each value of $x \in [0, \frac{\pi}{3}]$ we have a slice. Each such slice determines a cylindrical shell in the solid $\mathcal{S}$. The height of this shell is $\cos(x)$ and the radius is equal to x. Therefore, the volume of $\mathcal{S}$ is equal to:

$$\int_0^{\frac{\pi}{3}} 2\pi x \cos(x)\, dx$$

We can use integration by parts to compute this integral. Define the parts by $u = 2\pi x$ and $dv = \cos(x)dx$. Therefore, we have $du = 2\pi dx$ and $v = \sin(x)$:

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} 2\pi x \cos(x)\, dx &= 2\pi x \sin(x)|_0^{\frac{\pi}{3}} - \int_0^{\frac{\pi}{3}} 2\pi \sin(x)\, dx \\
&= 2\pi \frac{\pi}{3} \sin(\frac{\pi}{3}) - 2\pi \times 0 \sin(0) - (-2\pi \cos(x)|_0^{\frac{\pi}{3}}) \\
&= 2\frac{\pi^2}{3}\frac{\sqrt{3}}{2} + 2\pi \cos(\frac{\pi}{3}) - 2\pi \cos(0) \\
&= \frac{\sqrt{3}\pi^2}{3} + 2\pi\frac{1}{2} - 2\pi \\
&= \frac{\sqrt{3}\pi^2}{3} - \pi
\end{aligned}$$

(c) Let $\mathcal{T}$ be the solid given by rotating the region $\mathcal{R}$ about the horizontal line $y = 2$. Find the volume of $\mathcal{T}$.

Solution. We slice the region $\mathcal{R}$ vertically again and for each $x \in [0, \frac{\pi}{3}]$ we obtain one slice. However, such slice in this case gives rise to a washer because we are rotating a vertical slice about a horizontal line. The inner radius of each slice $2 - \cos(x)$ and the outer radius is equal to 2. Therefore, volume of a slice with thickness Δx at the point $x \in [0, \frac{\pi}{3}]$:

$$(\pi 2^2 - \pi(2 - \cos(x))^2)\Delta x$$

Therefore, the volume of the solid is equal to:

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} (\pi 2^2 - \pi(2-\cos(x))^2)\,dx &= \int_0^{\frac{\pi}{3}} (4\pi - \pi(4 - 4\cos(x) - \cos(x)^2))\,dx \\
&= \int_0^{\frac{\pi}{3}} 4\pi\cos(x) - \pi\cos(x)^2\,dx \\
&= 4\pi \int_0^{\frac{\pi}{3}} \cos(x)\,dx - \pi \int_0^{\frac{\pi}{3}} \cos(x)^2\,dx \\
&= 4\pi \sin(x)|_0^{\frac{\pi}{3}} - \pi \int_0^{\frac{\pi}{3}} \frac{1+\cos(2x)}{2}\,dx \\
&= 4\pi\sin(\frac{\pi}{3}) - 4\pi\sin(0) - \frac{\pi}{2}\int_0^{\frac{\pi}{3}} 1\,dx - \frac{\pi}{2}\int_0^{\frac{\pi}{3}} \cos(2x)\,dx \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi}{2}\frac{\pi}{3} - \frac{\pi}{2}\int_{2\times 0}^{2\times\frac{\pi}{3}} \cos(u)\,\frac{du}{2} \qquad (4) \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi^2}{6} - \frac{\pi}{4}(\sin(u)|_0^{\frac{2\pi}{3}}) \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi^2}{6} - \frac{\pi}{4}(\sin(\frac{2\pi}{3}) - \sin(0)) \\
&= 4\pi\frac{\sqrt{3}}{2} - \frac{\pi^2}{6} - \frac{\pi}{4}\frac{\sqrt{3}}{2} \\
&= 15\pi\frac{\sqrt{3}}{8} - \frac{\pi^2}{6}
\end{aligned}$$

In step (4), we use integration by substitution with $u = 2x$