Calculus II

Midterm 2 Fall 2018

Name: ______________________ **ID:** ______________________

Instructions:

(1) Fill in your name and Columbia University ID at the top of this cover sheet.

(2) This exam is closed-book and closed-notes; no calculators, no phones.

(3) Leave your answers in exact form (e.g. $\sqrt{2}$, not ≈ 1.4) and simplify them as much as possible (e.g. 1/2, not 2/4) to receive full credit.

(4) Answer all questions in the space provided. If you need more room use the blank backs of the pages.

(5) Show your work; correct answers alone will receive only partial credit.

(6) This exam has 5 extra credit points.

Problem	1 (10 pts)	2 (10 pts)	3 (10 pts)	4 (10 pts)	5 (15 pts)	6 (15 pts)	7 (15 pts)	8 (20 pts)	Total (105 pts)
Score									

1. (I) (2 points) Which one of the following options is correct?

(a) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ are both convergent.

(b) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ are both divergent.

(c) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

(d) The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is divergent and the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is convergent.

Solution. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent to 0. On the other hand, the series $\sum_{n=1}^{\infty}\frac{1}{n}$, which is a p-series with $p = 1$, is divergent. Therefore, choice (c) is correct.

(II) (2 points) Of the following series listed below, select ALL which are geometric series. (There are exactly two correct answers.)

(a) $\sum_{n=1}^{\infty}\frac{e^n}{n}$ (b) $\sum_{n=1}^{\infty}2^{\frac{1}{n}}$ (c) $\sum_{n=0}^{\infty}\frac{3^{n+1}}{4^{n-1}}$ (d) $\sum_{n=1}^{\infty}n^{\frac{1}{2}}$ (e) $\sum_{n=1}^{\infty}e^{2n+5}$

Solution. Choice (c) and (e) are geometric series with common ratios $\frac{3}{4}$ and e^2.

(III) (3 points) Give an example of a divergent series $\sum_{n=1}^{\infty}a_n$ such that $\sum_{n=1}^{\infty}a_n^2$ is convergent. Explain briefly why your series satisfies these two conditions.

Solution. If we pick $a_n = \frac{1}{n}$, then the series $\sum_{n=1}^{\infty}a_n$ is a p-series with $p = 1$ and hence it is divergent. In this case, the series $\sum_{n=1}^{\infty}a_n^2$ is a p-series with $p = 2$. Therefore, it is convergent.

(IV) (3 points) Write the number $3.\overline{48} = 3.484848\ldots$ as the ratio of two integer numbers in a reduced form. (Just give a fraction as the final answer. You do not need to justify your answer.)

Solution. We have:

$$\begin{aligned} 3.484848\ldots &= 3 + \frac{48}{100} + \frac{48}{100^2} + \frac{48}{100^3} + \ldots \\ &= 3 + \frac{48}{100}(1 + \frac{1}{100} + \frac{1}{100^2} + \ldots) \\ &= 3 + \frac{48}{100}(\frac{1}{1 - \frac{1}{100}}) \\ &= 3 + \frac{48}{99} \\ &= 3 + \frac{16}{33} \\ &= \frac{115}{33} \end{aligned}$$

Determine whether the following three series converge or not. State the tests that you are using and show your work. (10 points for each series)

2. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$

Solution. We apply the integral test to $f(x) = \frac{1}{x(\ln(x))^2}$. Observe that when $x \geq 3$, f is non-negative, decreasing and continuous. Moreover, we have:

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x(\ln(x))^2}\,dx &= \lim_{t\to\infty} \int_3^{t} \frac{1}{x(\ln(x))^2}\,dx \\
&= \lim_{t\to\infty} \int_{\ln(3)}^{\ln(t)} \frac{1}{u^2}\,du \qquad \text{substitution using } u = \ln(x) \\
&= \lim_{t\to\infty} (-\frac{1}{u})\Big|_{\ln(3)}^{\ln(t)} \\
&= \lim_{t\to\infty} \frac{1}{\ln(3)} - \frac{1}{\ln(t)} \\
&= \frac{1}{\ln(3)}
\end{aligned}$$

Therefore, by the Integral test, the series $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ is also convergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3}{2n^3+3n-1}$

Solution. Using the divergence test (the n^{th}-term test): We can use the divergence test to show that the series is divergent The limit of the terms of the series can be computed as

$$\lim_{n\to\infty} \frac{3n^3}{2n^3+3n-1} = \lim_{n\to\infty} \frac{3\frac{n^3}{n^3}}{2\frac{n^3}{n^3}+3\frac{n}{n^3}-\frac{1}{n^3}} = \lim_{n\to\infty} \frac{3}{2+3\frac{1}{n^2}-\frac{1}{n^3}} = \frac{3}{2}$$

Since this limit is not equal to 0, the series is divergent by the divergence test.

Using limit comparison Test: We can use limit comparison test to show the divergence of this series. Let $\displaystyle\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3}{2}$. Then:

$$\lim_{n\to\infty} \frac{\frac{3n^3}{2n^3+3n-1}}{\frac{3}{2}} = \frac{2}{3}\lim_{n\to\infty} \frac{3\frac{n^3}{n^3}}{2\frac{n^3}{n^3}+3\frac{n}{n^3}-\frac{1}{n^3}} = \frac{2}{3}\lim_{n\to\infty} \frac{3}{2+3\frac{1}{n^2}-\frac{1}{n^3}} = 1 \qquad (1)$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{3}{2}$ is divergent, because the terms of the series are constant and non-zero. Since the limit in (**??**) is not equal to 0 or ∞, the limit comparison test implies that the series is divergent.

4. $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+2}{n^4}}$

Solution. Using comparison Test: We have $n+2 \leq 3n$ for all $n \geq 1$. Therefore, for $n \geq 1$, we can write:

$$n+2 \leq n+2n = 3n \implies$$

$$0 \leq \sqrt{\frac{n+2}{n^4}} \leq \sqrt{\frac{3n}{n^4}} = \sqrt{3}\frac{1}{n^{\frac{3}{2}}}$$

Since $\displaystyle\sum_{n=1}^{\infty} \sqrt{3}\frac{1}{n^{\frac{3}{2}}} = \sqrt{3}\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ is a multiple of a p-series with $p = \frac{3}{2}$, it is convergent. Therefore, we can apply the comparison test to conclude that $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+2}{n^4}}$ is also convergent.

Using Limit comparison Test: We can use limit comparison test with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ to show that our series is convergent:

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n+2}{n^4}}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \sqrt{\frac{n+2}{n}} = \lim_{n\to\infty} \sqrt{1+\frac{2}{n}} = 1$$

Since this limit is between 0 and ∞, the series $\displaystyle\sum_{n=1}^{\infty} \sqrt{\frac{n+2}{n^4}}$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges. The latter series is a p-series with $p = \frac{3}{2}$. Therefore, it is convergent.

5. (15 points) For each of the following improper integrals, determine whether it is convergent or not. If it is convergent, evaluate the integral:

(a) $\displaystyle\int_1^\infty xe^{-x^2}\,dx$

Solution. We use the substitution $u = x^2$ to compute this improper integral. Note that we have $du = 2xdx$.

$$\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\int_1^{t^2}\frac{1}{2}e^{-u}\,du \\
&= \lim_{t\to\infty}\frac{1}{2}(-e^{-u})|_1^{t^2} \\
&= \lim_{t\to\infty}\frac{1}{2}(e^{-1}-e^{-t^2}) \\
&= \frac{1}{2}e^{-1}
\end{aligned}$$

(b) $\displaystyle\int_0^1 \frac{2+\sin(x)}{x^3}\,dx$

Solution. We use comparison to show that this improper integral is divergent. Since $-1 \leq \sin(x) \leq 1$ for all x, we have:

$$\frac{2+\sin(x)}{x^3} \geq \frac{2-1}{x^3} \geq \frac{1}{x^3}$$

$$\begin{aligned}
\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\int_1^{t^2}\frac{1}{2}e^{-u}\,du \\
&= \lim_{t\to\infty}\frac{1}{2}(-e^{-u})|_1^{t^2} \\
&= \lim_{t\to\infty}\frac{1}{2}(e^{-1}-e^{-t^2}) \\
&= \frac{1}{2}e^{-1}
\end{aligned}$$

Since the p-integral $\int_0^1 \frac{1}{x^3}\,dx$ is divergent, the integral $\displaystyle\int_0^1 \frac{2+\sin(x)}{x^3}\,dx$ is also divergent by comparison test.

6. (a) (5 points) Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$. We use the sum of the first 10 terms to approximate the sum of this series. Estimate the error involved in this approximation.

Solution. According to alternating series estimation theorem, the error is less than the absolute value of the first unused term, which is equal to $\frac{1}{11^3}$.

(b) (5 points) How many terms are required to ensure that the sum is accurate to three decimal places.

Solution. According to alternating series estimation theorem, if we use the sum of the first n terms to estimate $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^3}$, then the error is less than the absolute value of the first unused term, which is equal to $\frac{1}{(n+1)^3}$. We want the error to be less than $\frac{1}{1000}$. Therefore, it suffices to take $n = 9$.

(c) (5 points) Consider the series $\sum_{n=1}^{\infty}\frac{1}{n^3}$. We use the sum of the first 5 terms to approximate the sum of this series. Estimate the error involved in this approximation.

Solution. According to the remainder estimate for the integral test, we have:

$$\begin{aligned} e_5 &< \int_5^{\infty}\frac{1}{x^3} \\ &= \lim_{t\to\infty} -\frac{1}{2x^2}|_5^t \\ &= \lim_{t\to\infty}\frac{1}{50}-\frac{1}{2t^2} \\ &= \frac{1}{50} \end{aligned}$$

Therefore, e_5, the error of the sum of the first 5 terms to approximate the sum of the series, is less than $\frac{1}{50}$.

7. (a) (7 points) Find the length of the curve $y = \int_0^x \sqrt{t^2+6t+8}\,dt$ for $1 \le x \le 4$.

Solution. Using the fundamental theorem of calculus, we have $\frac{dy}{dx} = \sqrt{x2+6x+8}$. Therefore, the arc length formula tells us that:

$$\begin{aligned}
\text{Length of the given curve} &= \int_1^4 \sqrt{1+(\sqrt{x^2+6x+8})^2}\,dx \\
&= \int_1^4 \sqrt{x^2+6x+9}\,dx \\
&= \int_1^4 \sqrt{(x+3)^2}\,dx \\
&= \int_1^4 x+3\,dx \\
&= \frac{x^2}{2} + 3x|_1^4 \\
&= (8+12) - (\frac{1}{2}+3) \\
&= \frac{33}{2}
\end{aligned}$$

(b) (8 points) Find the area of the surface obtained by rotating the curve $x = 2+3y^2$, for $2 \le y \le 3$ about the x-axis.

Solution. We can use the area formula for a surface of revolution associated to a curve of the form $x = g(y)$ to write:

$$\begin{aligned}
\text{Area of the given surface} &= 2\pi \int_2^3 y\sqrt{1+\frac{dx}{dy}^2}\,dy \\
&= 2\pi \int_2^3 y\sqrt{1+(6y)^2}\,dy \qquad\qquad u = 1+(6y)^2,\ du = 72y \\
&= 2\pi \int_{1+(6\times 2)^2}^{1+(6\times 3)^2} \sqrt{u}\,\frac{du}{72} \\
&= \frac{2\pi}{108} u^{\frac{3}{2}}|_{145}^{325} \\
&= \frac{\pi}{54}(325^{\frac{3}{2}} - 145^{\frac{3}{2}})
\end{aligned}$$

8. (20 points) A swimming pool, filled with water, has the shape of an inverted frustum. (A frustum is obtained from a right circular cone by cutting off the tip.) The radius of the upper and lower bases are respectively equal to 12m and 8m. The height of the pool is also equal to 2m. Find the work required to empty the pool by pumping all of the water to the top of the pool. (The density of water is 1000kg/m^3.)

Solution. We use a coordinate axis denoted by h to measure the depth from the top of the pool. (See the figure.)Therefore, the cross section of the pool at $h = 0$ (respectively, $h = 2$) is a circle of radius 12 (respectively, 8). More generally, let $r(h)$ be the radius of the cross section at depth h. Then $r(h)$ is a linear function whose values at $h = 0$, 2 are equal to 12, 8, respectively. Therefore:

$$\begin{aligned} r(h) &= \frac{8-12}{2-0}h + 12 \\ &= -2h + 12 \\ &= 2(-h+6) \end{aligned}$$

(We can also use similar triangles to find the above formula for $r(h)$. See Example 5 of Section 6.4 of the textbook.) Following the same strategy of finding approximation and taking limit as in Example 5 of Section 6.4 of the textbook, we can conclude that:

$$\begin{aligned} \text{work} &= \int_0^2 1000(\pi r(h)^2) 9.8h\, dh \\ &= 9800\pi \int_0^2 4(-h+6)^2 h\, dh \\ &= 39200\pi \int_0^2 h^3 - 12h^2 + 36h\, dh \\ &= 39200\pi(\frac{h^4}{4} - 4h^3 + 18h^2|_0^2) \\ &= 39200\pi \times 44 \\ &= 1724800\pi \end{aligned}$$