Lectures 1 & 2

1 What are Knots and Links?

Before giving the definition of the main objects of interest for us, we introduce some standard notations. The standard n-dimensional Euclidean space is denoted by $\mathbf{R}^n$. The standard *n-dimensional ball* B_n consists of points x in $\mathbf{R}^n$ with $|x| < 1$, where $|\cdot|$ denotes the standard Euclidean norm on $\mathbf{R}^n$. The n-dimensional sphere is defined to be the boundary of the $(n+1)$-dimensional ball, i.e., the set of points x in $\mathbf{R}^{n+1}$ with $|x| = 1$. Removing a point from S^n produces a space which is homeomorphic (in fact, diffeomorphic) to $\mathbf{R}^n$.

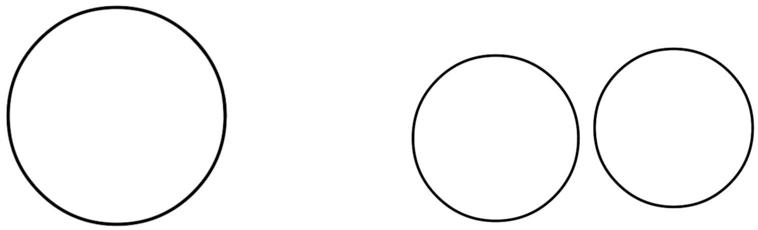
Definition 1.1. A **knot** K is a smooth embedding of S^1 into S^3. More generally, a **link** L with n connected components is an embedding of $\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{n}$ into S^3. A link is oriented if an orientation for each of its connected components is fixed.

Definition 1.2. We say two knots $K,\ K' : S^1 \to S^3$ are equivalent to each other, if there is an orientation-preserving smooth map $h : S^3 \to S^3$ such that $K' = h \circ K$. More generally, equivalence of two links with n connected components is defined similarly.

Remark 1.3. According to this definition, a knot or a link is a map. However, we often think about a knot or a link as the image of such maps, i.e., a subset of S^3. One way to justify this is to notice that knots with the same image are equivalent to each other. Soon we also do not distinguish between a knot and its equivalence class with respect to the relation given in Definition 1.2. Since we are concerned with the equivalence class of links, there is also no harm in replacing S^3 with $\mathbf{R}^3$.

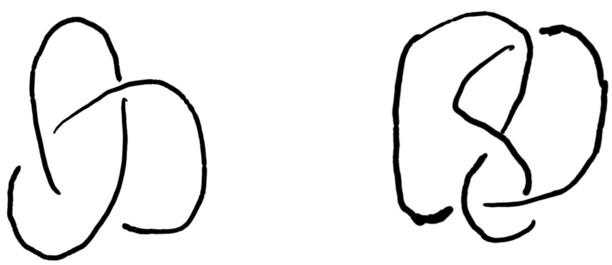
Remark 1.4. We defined knots, links and the equivalence of such objects in the smooth category. Alternatively, we can work in the *PL (piecewise-linear) category*. In this category, a knot K is a piecewise linear closed curve in S^3. Two knots K and K' are PL-equivalent if there is an orientation preserving homeomorphism $h : S^3 \to S^3$ which maps K to K'. The corresponding concepts for links with more than one connected component is defined similarly. There is a one-to-one correspondence between the set of PL knots modulo PL equivalences and set of smooth knots modulo smooth equivalence. We do not give the proof of this fact here. But note that a similar claim does not hold in higher dimensions. (See, for example, [Hae62].)

Both smooth and PL perspectives are useful in knot theory. Working in PL category gives a more combinatorial flavor to knot theory, and algorithmic questions are more accessible in this category. On the other hand, smooth category is more suitable for geometric methods in knot theory (e.g. hyperbolic geometry and several versions of knot Floer homology theories). As far as this course is concerned,



(a) (b)

Figure 1



(a) (b)

Figure 2

working in either category works for the most part. We usually make the choice that makes the exposition easier.

Example 1.5. The simplest knot is the *unknot* U given in Figure 1a. A PL representative for the unknot is given by the boundary of a triangle embedded in $\mathbf{R}^3$. Analogously, we may define the *unlink* with n connected components as the boundary of three disjoint triangles embedded in $\mathbf{R}^3$. Figure 1b shows an unlink with two connected components.

Example 1.6. Figures 2a and 2b show the two simplest knots after the unknot which are called trefoil and the figure-eight knot.

There are several other ways to define equivalence relation between knots. Here we give the definitions of other equivalence relations:

Definition 1.7. Suppose $K \subset \mathbf{R}^3$ is a link (in the PL sense) and T is an embedded triangle in $\mathbf{R}^3$ such that the boundary of T consists of the edges e_1, e_2, e_3. We assume that e_1 is one of the line segments of K and T does not intersect K otherwise. Let K' be the link which is obtained from K by removing e_1 and adding $e_2 \cup e_3$. Then we say K' is related to K by a Δ-move. In general, two links K and K' are called Δ-equivalent, if there is a sequence of Δ-moves and inverse of Δ-moves which relates K to K'.

Definition 1.8. Suppose $K, K' \subset \mathbf{R}^3$ are two links (in the PL sense). An ambient isotopy from K to K' is a homeomorphism $H : [0,1] \times \mathbf{R}^3 \to [0,1] \times \mathbf{R}^3$ (again in the PL sense) such that it satisfies the following properties:

(i) $H(t, x) = (t, h_t(x))$;

(ii) $h_0(x) = x$;

(iii) h_1 maps K to K'.

Proposition 1.9. ([BZ03, Proposition 1.10]) *The following properties for two links K and K' (in the PL sense) are equivalent to each other:*

(i) K and K' are equivalent to each other;

(ii) K and K' are ambient isotopic to each other;

(iii) K and K' are Δ-equivalent to each other.

Definition 1.10. Given two oriented knots K and K', the *connected sum* of K and K', denoted by $K\#K'$, is an oriented knot defined as follows. We regard K and K' as subsets of two distinct copies of S^3 and fixed embedded balls B and B' in these two S^3's such that $B \cap K$ and $B' \cap S^3$) are unknotted arcs. After removing B and B' from these spheres and choosing an orientation reversing identification of the boundary components ∂B and $\partial B'$ that maps $K \cap (\partial B)$ to $K' \cap (\partial B')$, we obtain another copy of S^3 and a knot $K\#K'$ in this new 3-sphere. Orientations on K and K' induce orientations on $K\#K'$ and we require that the identification of ∂B and $\partial B'$ is chosen such that these two orientations on $K\#K'$ are compatible with each other. The equivalence class of $K\#K'$ is independent of the chosen balls and the identification of their boundaries.

Definition 1.11. Connected sum $K\#U$ of a knot K and the unknot U is equivalent to K. We say K is *prime* if any presentation of K as a connected sum $K_1\#K_2$ implies that either K_1 or K_2 is the unknot.

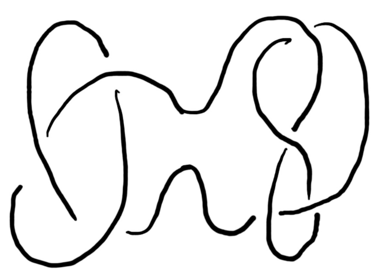


Figure 3: This figure shows the connected sum of the trefoil and the figure-eight knot.

One of the early questions in knot theory was to classify prime knots with a small number of crossings[1]. As an example, see Figure 8 which gives a classification of such knots by Tait in 1884. There are some mistakes in this table which was subsequently fixed.

[1]See Section 2 for the definition of crossing number for a knot K.

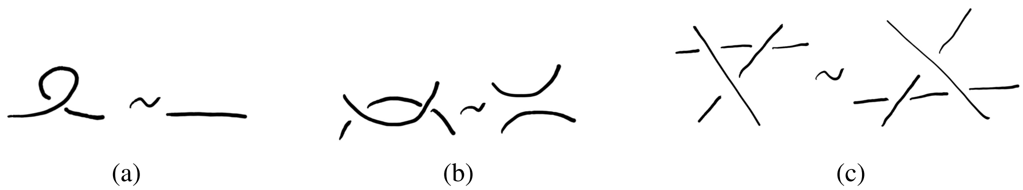


Figure 4: The above moves and their inverses are called Reidemeister moves.

2 Link Diagrams

Link diagrams provide a helpful way of representing links. Suppose $K \subset \mathbf{R}^3$ is a link (in the PL sense). In order to form a diagram we consider the projection map $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ along the z-axis given by $\pi(x, y, z) = (x, y)$. We say K is in *general position* with respect to π if the inverse image of each point in $\mathbf{R}^2$ intersects the link K in at most two points. Furthermore, if a vertex point of K belongs to this inverse image, then the inverse image contains exactly one point. We can always find a representative for K which is in general position with respect to π by applying Δ-moves and their inverses.

Definition 2.1. Suppose a representative for K is given which is in general position with respect to π. Then $\pi(K)$ determines a closed curve in $\mathbf{R}^2$ with a number of double points. Any double point of $\pi(K)$ is called a *crossing*. At each crossing, two line segments of $\pi(K)$ intersect. The segment which goes over is called an *over-crossing* and the other one is called an *under-crossing*. The curve $\pi(K)$ and the information of over-crossing and under-crossing at each crossing is called a *diagram* for K.

Any diagram D of K determines (the equivalence class of) K. However, a link K has more than one diagram. For example, we can change a diagram of K by the moves in Figure 4 or their inverses. These moves are called *Reidemeister moves*. The following theorem asserts that these moves would be sufficient to obtain all diagrams of K. This theorem can be proved by analyzing what happens to the diagram of K after a Δ-move.

Theorem 2.2. *Two knots are equivalent if and only if their diagrams are equivalent to each other by a sequence of Reidemeister moves.*

One of the themes of knot theory (and in particular this class) is to define link invariants and use them to study links. We can use the notion of diagrams to define the first such invariant:

Definition 2.3. The crossing number of a link K is the minimum number of crossings among all diagrams for K.

In general, computing crossing number is not easy, and we shall develop more approachable invariants throughout the semester. See, however, Theorem 3.2, which we shall prove later.

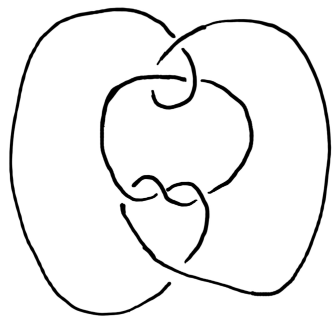


Figure 5: This is a non-trivial diagram for the unknot.

3 Some Families of Knots

Definition 3.1. A knot is *alternating* if it admits a link diagram such that the over-crossings and under-crossings alternates as we travel along the knot. The trefoil and the figure eight knot are both alternating.

Theorem 3.2. *The number of crossings for a reduced alternating diagram of a knot K is equal to the crossing number of K.*

The definition of reduced diagrams and the proof of this theorem will be given later in the class.

As two other families of knots, see Figures 6 and 7. Figure 6 sketches the (p, q)-torus link usually denoted by $T_{p,q}$. In the case that p and q are coprime, $T_{p,q}$ is a knot. Figure 7a gives a schematic picture of a Pretzel link $P(a_1, \ldots, a_n)$. Here a block with an integer a_i consists of two strands which is twisted to the left a_i times if a_i is positive and is twisted to the right $-a_i$ times if a_i is negative. (See Figure 7b)

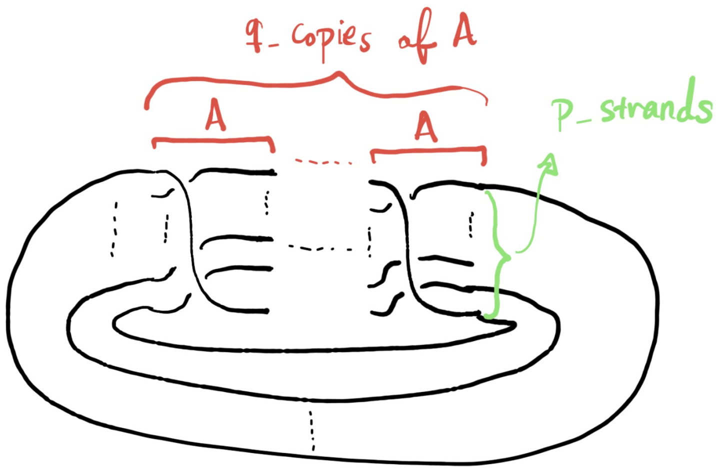
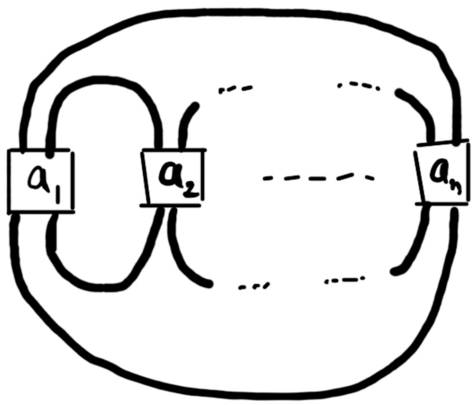
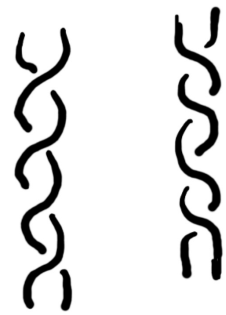


Figure 6: Torus Links



(a) A schematic figure of Pretzel link



(b) The strand on the left represent 4 left-handed twists and the strands on the right represent 4 right-handed twists

Figure 7: Pretzel link

4 Possible Topics

Here is a list of topics that I plan to cover in this class. This list will be updated based on students' interests:

(i) Jones polynomial;

(ii) Seifert surfaces;

(iii) Cyclic branched covers of S^3 along knots;

(iv) Invariants obtained from cyclic covers: Alexander polynomial, signature, etc.;

(v) Representing 3-manifolds using links;

(vi) Studying fundamental groups of knot complements.

References

[BZ03] Gerhard Burde and Heiner Zieschang, *Knots*, Second, De Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 2003. MR1959408 ↑3

[Hae62] André Haefliger, *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math. (2) **75** (1962), 452–466. MR0145539 ↑1

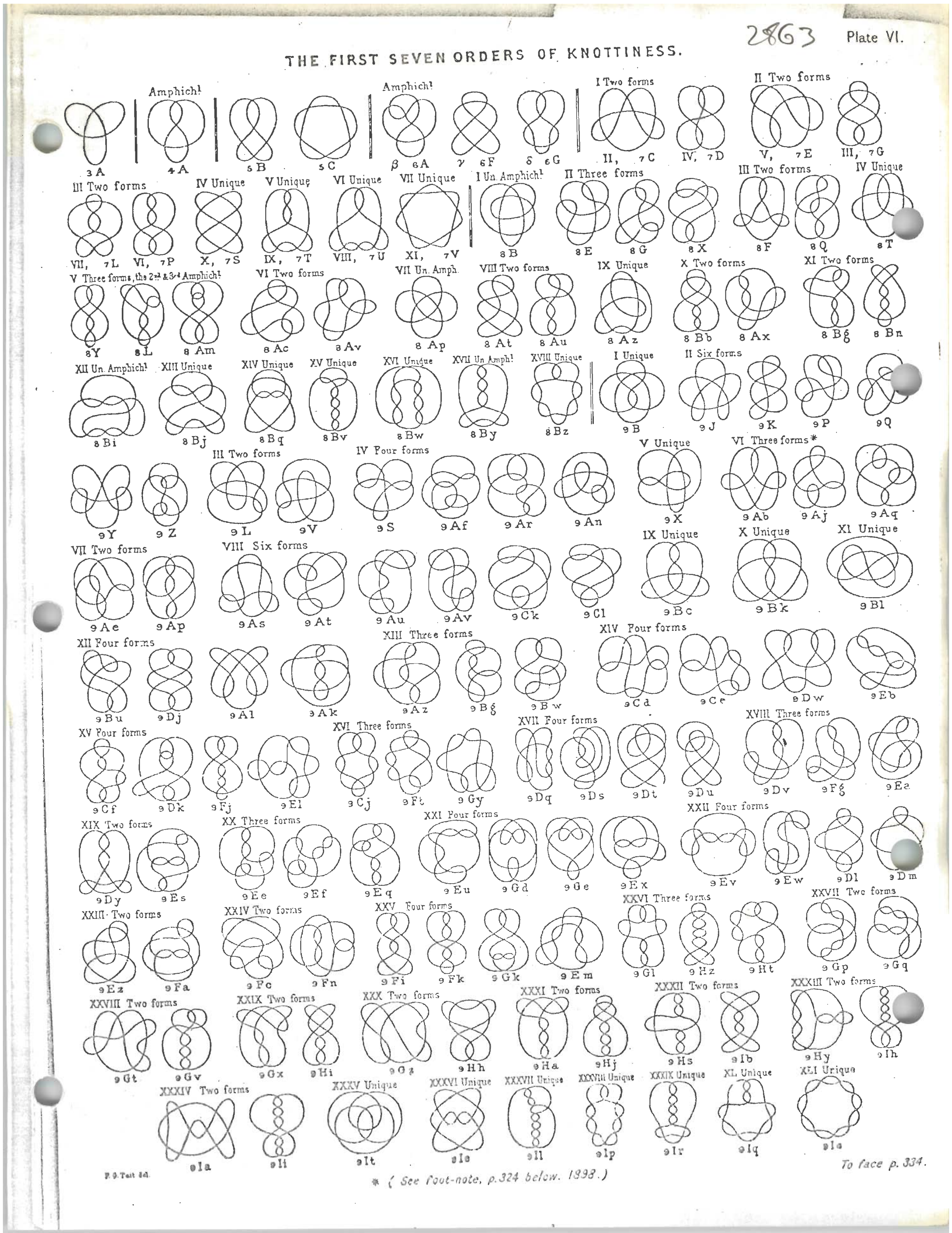


Figure 8: Classification of knots with Small number of Crossings by Tait in 1884