Lecture 10

1 Introduction

In this lecture, we state classification of orientable surfaces. To that end, we firstly explain one way of defining the notion of orientability for manifolds. Next, we define Seifert surfaces for oriented links, which are very useful objects in knot theory.

2 Orientations of Smooth Manifolds

Suppose V is an n-dimensional real vector space and $(f_1, \dots, f_n)$ is a basis for V. If $(e_1, \dots, e_n)$ is another basis for V, then there is a unique element $A = (A_i^j)$ of $\mathrm{GL}(n, \mathbf{R})$ such that:

$$e_i = \sum_j A_i^j f_j.$$

We partition the set of bases of V, denoted by $\mathfrak{B}$, into two subsets $\mathfrak{B}_0$ and $\mathfrak{B}_1$ where $(e_1, \dots, e_n) \in \mathfrak{B}_0$ if and only if the determinant of A is positive. It is easy to see that this partition is independent of the choice of the basis $(f_1, \dots, f_n)$ up to relabeling the two sets. An orientation for V is a choice of one of the sets $\mathfrak{B}_0$ and $\mathfrak{B}_1$. Any basis element in the chosen set is called a *positively oriented* basis and any basis element in the other set is a *negatively oriented* basis.

Example 2.1. The following basis for $\mathbf{R}^n$

$$e_1 = (1, 0, \dots, 0), \qquad e_2 = (0, 1, \dots, 0), \qquad \dots \qquad e_n = (0, 0, \dots, 1)$$

represents the standard orientation. The other orientation of $\mathbf{R}^n$ is represented by $(e_2, e_1, \dots, e_n)$.

If M is a smooth manifold of dimension n and $p \in M$, then the tangent space of M at p gives an n-dimensional vector space. This space can be thought as the vector space of tangent vectors at $t = 0$ of curves $\gamma(t)$ such that $\gamma(0) = p$. A choice of orientations for all tangent spaces at points p of M such that the orientations depend continuously on p is called an *orientation* of M. We say M is *orientable* if it admits an orientation. If M is path connected and orientable, then there are exactly two choices of orientations.

Example 2.2. Any Euclidean space $\mathbf{R}^n$ is orientable.

Example 2.3. If M is an orientable manifold with boundary, then it determines an orientation on its boundary ∂M. Let p be a point on the boundary of M. Then the tangent space to M at the point is given by the sum of the tangent space to ∂M and the 1-dimensional space spanned by the outward pointing normal vector ν. If $(e_1, \ldots, e_n)$ is a basis of the tangent space of ∂M at p, then we declare that this basis is positively oriented if $(\nu, e_1, \ldots, e_n)$ is a positively oriented basis for M. Since S^n is the boundary of the $(n+1)$-dimensional ball D^{n+1}, the n-dimensional sphere is orientable.

Example 2.4. If M and N are two orientable manifolds, then their products $M \times N$ is also orientable. The vectors tangent to a point $(p, q) \in M \times N$ can be identified with the direct sum of the space of vectors tangent to M at the point p and the space of vectors tangent to N at the point q. In particular, if $(e_1, \ldots, e_m)$ represents a choice of an orientation of M at p and $(e'_1, \ldots, e'_n)$ represents a choice of an orientation of N at q, then $(e_1, \ldots, e_m, e'_1 \ldots, e'_n)$ determines an orientation of $M \times N$ at the point (p, q).Using this construction, we can verify orientability of $M \times N$ assuming orientability of M and N. In particular, the 2-dimensional torus $S^1 \times S^1$ is orientable.

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Example 2.6. The Möbius strip is not orientable.

3 Classification of Orientable Surfaces

For our purposes in this class, a *surface* is a compact, connected and orientable 2-dimensional manifold (possibly with boundary). Any surface admits a unique smooth structure and hence we can focus on smooth surfaces. The 2-dimensional sphere S^2 and the torus $S^1 \times S^1$ are examples of surfaces. More generally, the spaces Σ_g and Σ_g°, introduced in the previous lectures, are examples of surfaces.

We firstly give a classification of orientable surfaces with empty boundary. Any such surface is diffeomorphic to Σ_g for an appropriate choice of g. Therefore, orientable surfaces (without boundary) is determined by the genus number g. Recall that the homology groups of Σ_g are given as follows:

$$H_0(\Sigma_g) = H_2(\Sigma_g) = \mathbf{Z}, \qquad H_1(\Sigma_g) = \mathbf{Z}^{2g}.$$

In general, for a topological space we define the Euler characteristic of a topological space X to be:

$$e(X) := \mathrm{rank}(H_0(X)) - \mathrm{rank}(H_1(X)) + \mathrm{rank}(H_2(X)) - \mathrm{rank}(H_3(X)) + \ldots$$

Here we assume that the homology group $H_i(X)$ vanishes when i is large enough. Our calculation above shows that the Euler characteristic of Σ_g is equal to $2 - 2g$. Therefore, we can also conclude that any orientable surface with empty boundary is determined by its Euler characteristic.

Next, we recall classification of all orientable surfaces with boundary. Let $\Sigma_{g,d}$ be the surface obtained by removing the interior of d open discs from Σ_g. (See Figure 1.) In particular, $\Sigma_{g,1}$ is the surface that we denote by Σ_g°. The boundary of the surface $\Sigma_{g,d}$ has d connected components. An easy application of Mayer-Vietoris sequence implies that for $d \geqslant 1$:

$$H_0(\Sigma_g) = \mathbf{Z}, \qquad H_1(\Sigma_g) = \mathbf{Z}^{2g+d-1}, \qquad H_2(\Sigma_g) = 0.$$

In particular, the Euler characteristic of $\Sigma_{g,d}$ is equal to $2 - 2g - d$. (Notice that the same formula holds even if $d = 0$.) Any orientable surface M is diffeomorphic to $\Sigma_{g,d}$ for appropriate choices of g and d. The number d is equal to the number of the boundary components of M, and g is equal to $\frac{2-e(M)-d}{2}$.

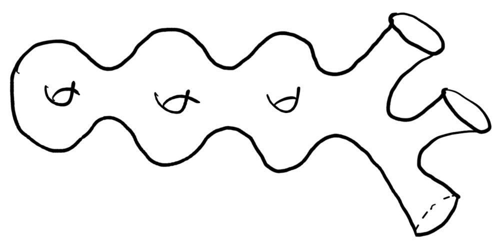

Figure 1: The surface $\Sigma_{3,3}$

4 Seifert Surfaces

Definition 4.1. Suppose L is an oriented link in S^3. An oriented surface Σ embedded in S^3 which has L as its oriented boundary is *Seifert surface* for the Riemann surface Σ.

Proposition 4.2. *Any oriented link L admits a Seifert surface.*

A priori, it might not be obvious that any L admits a Seifert surface. In fact, we can construct a Seifert surface for L staring with a diagram for D. Suppose $s_o(D)$ denotes the diagram of D obtained by resolving all crossings of D into the oriented resolution. That is to say, we resolve all positive crossings of D to positive resolutions and all negative crossings of D to negative resolutions. The resulting digram consists of an oriented digram for the unlink with $|s_o(D)|$ components. Each of the connected components of $s_o(D)$ is called a *Seifert circuit*. We can find a union of $|s_o(D)|$ oriented discs whose boundary is equal to $s_o(D)$. In order to obtain a surface whose boundary is equal to L, we glue a band as in Figure 2. This construction of Seifert surfaces is called Seifert algorithm.

Proposition 4.3. *Suppose D is a diagram for an oriented link L with m connected components. Suppose n denotes the number of crossings of D. Then Seifert algorithm applied to D produces a Seifert surface with Euler characteristic $|s_o(D)| - n$ and m boundary components. In particular, the genus of this Seifert surface is equal to $\frac{2+n-|s_o(D)|-d}{2}$.*

For the proof of this proposition, we need the following basic lemma about Euler characteristics, which is again a consequence of the Mayer-Vietoris sequence:

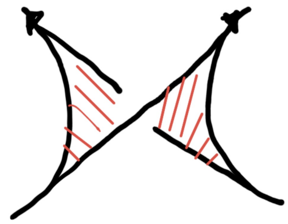

Figure 2

Lemma 4.4. *If U and V are open subspaces of X, then $e(X) = e(U) + e(V) - e(U \cap V)$.*

Proof of Proposition 4.3. The Euler characteristic of the union of discs which fill the Seifert circuits is equal to $|s_o(D)|$. Above lemma implies that adding each band subtracts 1 from the Euler characteristic. This observation verifies the claim. □

Suppose K is a knot and Σ is a Seifert surface for K. Suppose also $X(K)$ denotes an exterior of the knot K. We may assume that Σ intersects the boundary of $X(K)$ transversely by perturbing Σ. Then the intersection of Σ with the boundary of $X(K)$ is an oriented simple closed curve λ. Since λ bounds a surface embedded in $X(K)$, it represents a trivial element of $H_1(X(K))$. In particular, λ defines a longitude of the knot K.

The following observation is a consequence of the assumption that Seifert surfaces are oriented and will be useful later for us:

Lemma 4.5. *Suppose F is the intersection of a Seifert surface of K with its exterior. Then there is a neighborhood of F in $X(K)$ which is homeomorphic to $(-1, 1) \times F$.*

Proof. Since F is orientable, we can find a smooth family of vectors $v(x)$ for points $x \in F \subset S^3$ such that $v(x)$ is orthogonal to vectors which are tangent to F at the point x. Now we define a map $\Phi : (-\varepsilon, \varepsilon) \times F \to S^3$ where $\Phi(x, t)$ is equal to $x + t \cdot v(x)$. Clearly, $\Phi|_{\{0\}\times F}$ gives the Seifert surface F. Using inverse function theorem from real analysis, one can see that this map is a diffeomorphism. This verifies the claim. (To be more precise, we need to modify this map in a neighborhood of the boundary of $X(K)$.) □

5 Seifert Genus

Definition 5.1. *Seifert genus* of a knot K, denoted by $g(K)$, is the minimum among the genera of Seifert surfaces of K.

A knot is the unknot if and only if its Seifert genus is equal to 0. The following theorem asserts that Seifert genus is additive with respect to connected sum.

Theorem 5.2. *([Lic97, Theorem 2.4]) For any two knots K_1 and K_2, we have $g(K_1 \# K_2) = g(K_1) + g(K_2)$.*

References

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978 ↑5