Lectures 12-16

1 Introduction

The main purpose of this note is to define *Alexander polynomials* of a link K. As a first step toward the definition, we associates a topological space $X_\infty(K)$ to K with a homeomorphism $\Phi : X_\infty(K) \to X_\infty(K)$. The pair $(X_\infty(K), \Phi)$ defines a link invariant. (See Theorem 3.2.) By considering the first homology of $X_\infty(K)$ and the action of Φ on this homology group, we obtain an algebraic link invariant in the shape of a $\mathbf{Z}[t^{-1}, t]$-module. Form this module, we construct simpler algebraic invariants In Section 4. In Section 5, we introduce the notion of *Seifert forms* and explain how it can be used to compute the algebraic invariants introduced in the earlier sections.

2 Regular Neighborhoods of Embedded Surfaces in R^3

Suppose Σ is a Seifert surface for an oriented link in S^3. We would like to see how a neighborhood of Σ looks like. We firstly start with a simpler case:

Lemma 2.1. *Suppose F is an embedding of the closed surface Σ_g in S^3. Then there is a neighborhood of F in $X(K)$ which is homeomorphic to $(-1, 1) \times \Sigma_g$.*

Proof. Firstly we claim that orientability of F can be used to construct $v : F \to \mathbf{R}^3$ such that for any $x \in F$, the vector $v(x)$ is orthogonal to vectors to F at the point x. In fact, for each x there are exactly two vectors v_1 and $-v_1$ which have length 1 and are tangent to the tangent space of F at the point x. Let e_1 and e_2 form an oriented basis for the tangent space of F at the point x. Then exactly one of (v_1, e_1, e_2) and $(-v_1, e_1, e_2)$ is an oriented basis for $\mathbf{R}^3$. In the former case, we define $v(x) = v_1$. Otherwise, $v(x) = -v_1$. Now we define a map $\Phi : (-\varepsilon, \varepsilon) \times F \to S^3$ where $\Phi(x, t)$ is equal to $x + t \cdot v(x)$. Clearly, $\Phi|_{\{0\}\times F}$ gives the embedded surface F. Using the inverse function theorem from real analysis, one can see that if ε is small enough, then this map is a diffeomorphism. This verifies the claim. $\square$

Lemma 2.2. *Suppose Σ is an embedding of the surface $\Sigma_{g,d}$ in S^3. Then there is a neighborhood $\mathcal{U}_\Sigma$ of Σ in S^3 which has the form $(-1, 1) \times \Sigma_{g,d}$. Moreover, if we remove Σ from this neighborhood, then the resulting space is homeomorphic to $\Sigma_{2g+d-1} \times (0, 1)$.*

Proof. The boundary of $\Sigma_{g,d}$ produces an oriented link K with d connected components in S^3. Let the link exterior $X(K)$ is obtained by removing a small regular neighborhood of K. Let also $N(K)$ be the open subset given as the complement of $X(K)$ in S^3. Then $N(K)$ is homeomorphic to d copies of $S^1 \times D^2$ and $\Sigma \cap N(K)$ is homeomorphic to d copies of $S^1 \times [0,1)$, where $[0,1)$ should be regarded as a ray from the center of D^2 to a point in the boundary.

A modification of the proof of the last lemma shows that a neighborhood of $\Sigma \cap X(K)$ is homeomorphic to $(-1,1) \times \Sigma_{g,d}$. (The modification needs to be made in a neighborhood of the boundary of $X(K)$.) Then the desired neighborhood $\mathcal{U}_\Sigma$of Σ is obtained by taking the union of $N(L)$ and the constructed neighborhood of $\Sigma \cap X(K)$. It is not hard to see that this neighborhood satisfies the required properties. $\square$

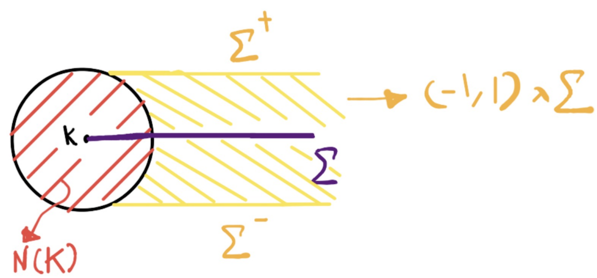


Figure 1: A schematic picture of the link K and the neighborhood $\mathcal{U}_\Sigma$ of Σ given as the union of red and yellow regions. In fact, if we focus on a normal plane to K, then the above picture is accurate.

3 The Space $X_\infty(K)$

Suppose K is an oriented link and Σ is a Seifert surface for K. As before let $X(K)$ be the exterior of K given by removing a small neighborhood of K. Then the intersection of Σ with $X(K)$ is again a surface which is homeomorphic to Σ and its boundary lies on the boundary of $X(K)$. With a slight abuse of notation, we denote this embedded surface in $X(K)$ with Σ, too. Using the proof of Lemma 2.2, we see that a neighborhood of Σ is diffeomorphic to $\Sigma \times (-1,1)$. After removing this neighborhood from $X(K)$ we obtain another 3-manifold with boundary $Y(K)$ such that its boundary is the union of three pieces:

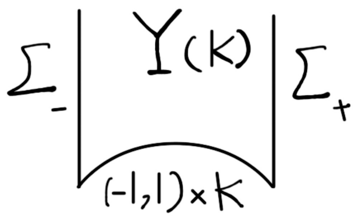
$$\partial Y(K) = \Sigma \times \{-1\} \cup K \times (-1,1) \cup \Sigma \times \{1\}$$

Note that the middle term is given by removing an annulus neighborhood $\Sigma \cap \partial X(K)$ from $\partial X(K)$. There are embeddings $\phi^- : \Sigma \to \partial Y(K)$ and $\phi^+ : \Sigma \to \partial Y(K)$ given by the first term and the third term in the above description of $\partial Y(K)$.

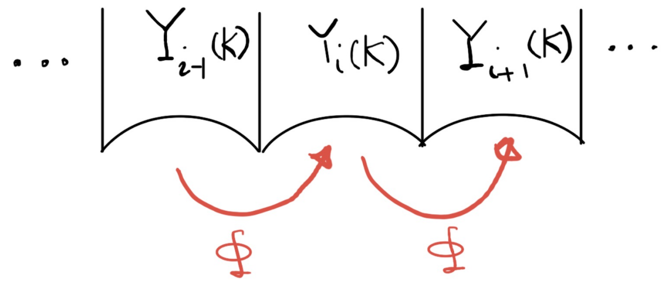
We wish to define a space $X_\infty(K)$ as a quotient of infinitely many copies of $Y(K)$. For each integer $i \in \mathbf{Z}$, let $Y_i(K)$ be a copy of $Y(K)$ and $\phi_i^- : \Sigma \to \partial Y_i(K)$ and $\phi_i^+ : \Sigma \to \partial Y_i(K)$ be the corresponding embeddings of Σ in $\partial Y_i(K)$. Then define:

$$X_\infty(K) = (\cdots \cup Y_{i-1}(K) \cup Y_i(K) \cup Y_{i+1}(K) \cup \dots)/\sim$$

where $\sim$ identifies $\phi_i^+(x)$ and $\phi_{i+1}^-(x)$ for any $i \in \mathbf{Z}$ and $x \in \Sigma$. That is to say, we glue each $Y_i(K)$ to $Y_{i-1}(K)$ and $Y_{i+1}(K)$ respectively along $\mathrm{image}(\phi_i^-)$ and $\mathrm{image}(\phi_i^+)$.



(a) The space $Y(K)$



(b) The space $X_\infty(K)$ together with the homeomorphism Φ

Figure 2: In these figures, the black (resp. red) components are part of the diagram D_1 (resp. D_2) for the link K_1 (resp. K_2).

There is an obvious homeomorphism $\Phi : X_\infty(K) \to X_\infty(K)$ which maps $Y_i(K)$ to $Y_{i+1}(K)$. This homeomorphism induces an isomorphism $t : H_1(X_\infty(K)) \to H_1(X_\infty(K))$. In particular, the abelian group $H_1(X_\infty(K))$ is equipped with the structure of a $\mathbf{Z}[t^{-1}, t]$-module.

Example 3.1. In the case that K is the unknot, we may take a disc to be the Seifert surface of K. In this case, $Y(K)$ is homeomorphic to $[-1, 1] \times D^2$ with $\phi^\pm : D^2 \to [-1, 1] \times D^2$ given by $\phi^\pm(x) = (\pm 1, x)$. In particular, the space $X_\infty(K)$ associated to K is given by $\mathbf{R} \times D^2$.

Theorem 3.2. *For any oriented link K, the space $X_\infty(K)$ together with the homeomorphism Φ is an invariant of K and does not depend on the choice of the Seifert surface Σ. In particular, the $\mathbf{Z}[t^{-1}, t]$-module $H_1(X_\infty(K))$ is an invariant of K.*

Later in the class, we prove this theorem using the theory of *covering spaces*. Before that we wish to answer the following questions:

1. What are some less complicated invariants that we can obtain from the $\mathbf{Z}[t^{-1}, t]$-module $H_1(X_\infty(K))$?
2. How can we give descriptions of the $\mathbf{Z}[t^{-1}, t]$-module $H_1(X_\infty(K))$?

4 Alexander Modules

Suppose R is a commutative ring with unit and M is an R-module. A finite presentation of M is an exact sequence of the following from:

$$F \xrightarrow{\alpha} E \xrightarrow{\pi} M \to 0$$

where E and F are free R-modules of finite ranks. In particular, the images of the basis elements of E under the map π determine a set of generators for M and any relation among these basis elements is a linear combination of the images of the basis elements of F under the map α. If $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ are the bases for E and F, the we define the *presentation matrix* $A = (a^i_j)^{1 \leqslant i \leqslant m}_{1 \leqslant j \leqslant n}$ of α as follows:

$$\alpha(f_j) = \sum a^i_j e_i.$$

Definition 4.1. For an R-module M with a finite presentation, let A be defined as above. Then the r-th *elementary ideal* $\mathcal{E}_r$ of M is defined to be the ideal generated by the minors of size $(m-r+1)\times(m-r+1)$ of A. In the case that $r > m$, $\mathcal{E}_r$ is defined to be R, and if $r \leqslant 0$, we define $\mathcal{E}_r = 0$.

Proposition 4.2. ([Lic97, Theorem 6.1.]) *The ideal $\mathcal{E}_r$ depends only on M and is independent of the choice of the finite presentation of M.*

For our purposes, the following algebraic facts are useful:

Lemma 4.3. *The ring $\mathbf{Z}[t^{-1}, t]$ is a UFD and $\mathbf{Q}[t^{-1}, t]$ is a PID.*

Definition 4.4. Given an oriented link K, the r-th elementary ideal $\mathcal{E}_r$ of the $\mathbf{Z}[t^{-1}, t]$-module $H_1(X_\infty(K))$ of M is called the r-th Alexander ideal of K. The r-th Alexander polynomial of K is a generator of the smallest principal ideal that contains $\mathcal{E}_r$. (Since $\mathbf{Z}[t^{-1}, t]$ is a UFD, this is well-defined.) Among these polynomial invariants of K, the first Alexander polynomial of K is the distinguished one, and usually is called the Alexander polynomial of K and is denoted by $\Delta_K(t)$.

Remark 4.5. Any unit in $\mathbf{Z}[t^{-1}, t]$ has the form $\pm t^n$ for an integer n. Therefore, at least given the above definition, the Alexander polynomials are well-defined up to multiplication by elements of the form $\pm t^n$.

5 Alexander Polynomials and Seifert Form

In this part, we give an answer to the second question raised in the end of Section 3.

5.1 Homology of the Space $Y(K)$

Fix a Seifert surface Σ for K as before, and let Σ be homeomorphic to $\Sigma_{g,d}$. We firstly attempt to compute the homology group $H_1(S^3\backslash\Sigma)$. Let $\mathcal{U}_\Sigma$ be an open neighborhood of Σ given by Lemma 2.2. Then $\mathcal{U}_\Sigma$ and $S^3\backslash\Sigma$ give an open covering of S^3 and their intersection is homeomorphic to $(0,1)\times\Sigma_{2g+d-1}$. The Mayer-Vietoris sequence implies that we have the following exact sequence:

$$\ldots \to H_2(S^3) \to H_1((0,1)\times\Sigma_{2g+d-1}) \xrightarrow{\Psi} H_1(S^3\backslash\Sigma) \oplus H_1(U_\Sigma) \to H_1(S^3) \to \ldots.$$

Since $H_1(S^3) = H_2(S^3) = 0$ and $(0,1)$ is contractible, we have the following natural isomorphism:

$$\Psi : H_1(\Sigma_{2g+d-1}) \xrightarrow{\cong} H_1(S^3\backslash\Sigma) \oplus H_1(U_\Sigma)$$

By Lemma 2.2, $\mathcal{U}_\Sigma$ is homeomorphic to $(-1,1)\times\Sigma_{g,d}$. In particular, $H_1(\Sigma) = \mathbf{Z}^{2g+d-1}$. Since $H_1(\Sigma_{2g+d-1}) = \mathbf{Z}^{2(2g+d-1)}$, we conclude that $H_1(S^3\backslash\Sigma)$ is also isomorphic to $\mathbf{Z}^{2g+d-1}$. It is easy to see that the space $Y(K)$ constructed in Section 3 has the same homotopy type as $S^3\backslash\Sigma$. Therefore, $H_1(Y(K))$ is also isomorphic to $\mathbf{Z}^{2g+d-1}$. Now that we obtained a good understanding of the homology groups $H_1(Y(K))$, we wish to study the homomorphisms $\phi^+_* : H_1(\Sigma) \to H_1(Y(K))$ and $\phi^-_* : H_1(\Sigma) \to H_1(Y(K))$ induced by continuous maps ϕ^+ and ϕ^-. To give a useful description for these maps, we need the notion of linking number, which is discussed in the next part.

5.2 Linking Number

Suppose K_1 and K_2 are two disjoint oriented knots in S^3. In particular, K_2 represents an element of the first homology of $X(K_1)$, the knot exterior K_1. Recall that $H_1(X(K_1))$ is isomorphic to **Z** and is generated by the homology class of a meridian of K_1. Therefore, K_2 is homologous to a multiple of the class of the meridian of K_1, and this multiple is called the *linking number* of K_1, K_2, and is denoted by $\mathrm{lk}(K_1, K_2)$. More generally, if K_1 (resp. K_2) is an oriented link with connected components $K_{1,1}, \ldots,$ $K_{1,m}$ (resp. $K_{2,1}, \ldots, K_{2,n}$), then:

$$\mathrm{lk}(K_1, K_2) := \sum_{1 \leqslant i \leqslant m,\, 1 \leqslant j \leqslant n} \mathrm{lk}(K_{1,i}, K_{2,j})$$

Remark 5.1. In Problem Set 3, we defined another notion of linking number using knot diagrams. These two definitions of linking number agree with each other. (See Problem Set 8.) It is clear from the definition in Problem Set 3 that $\mathrm{lk}(K_1, K_2) = \mathrm{lk}(K_2, K_1)$.

Now let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two disjoint connected open subsets of S^3. Then we can use the linking number to define a pairing:

$$\mathrm{lk} : H_1(\mathcal{U}_1) \times H_1(\mathcal{U}_2) \to \mathbf{Z}. \tag{5.2}$$

In fact, any homology class α_i of $\mathcal{U}_i$ can be represented by a simple closed curve K_i. Then $\mathrm{lk}(\alpha_1, \alpha_2)$ is defined to be the linking number of K_1 and K_2. It is clear from the above definition and Remark 5.1 that the linking number of two disjoint knots K_1 and K_2 do not change if we change the homology class of one of the knots while preserving the other component. Therefore, the pairing in (5.2) is well-defined.

5.3 Seifert Form

Now we go back again to the setup of Subsection 5.1. Let K be an oriented link and Σ be a Seifert surface for K. Suppose also a neighborhood $\mathcal{U}_\Sigma$ of Σ is given by Lemma 2.2. In particular, $\mathcal{U}_\Sigma$ is homeomorphic to $(-1, 1) \times \Sigma$. We consider the pairing lk for the pair $\mathcal{U}_1 = (-\frac{1}{3}, \frac{1}{3}) \times \Sigma$ and $\mathcal{U}_2 = S^3 \backslash [-\frac{1}{2}, \frac{1}{2}] \times \Sigma$. Since $\Sigma \subset \mathcal{U}_1$ and $Y(K) \subset \mathcal{U}_2$, the linking number determines a pairing as follows:

$$p : H_1(\Sigma) \times H_1(Y(K)) \to \mathbf{Z}$$

Lemma 5.3. *The pairing p is non-degenerate. That is to say, if $p(x, y) = 0$ for all choices of y (resp. x), then $x = 0$ (resp. $y = 0$).*

Proof. The surface Σ is homeomorphic to $\Sigma_{g,d}$. In particular, we may use the standard basis $e_1, e_2, \ldots,$ e_{2g+d-1} of $H_1(\Sigma_{g,d})$ to obtain a basis for $H_1(\Sigma)$. As it is shown in Figure 3, there are embedded paths $\gamma_1, \gamma_2, \ldots, \gamma_{2g+d-1}$ such that γ_i intersect e_i transversely at exactly one point and it is disjoint from e_j and γ_j for $j \neq i$. Moreover, γ_i intersect $\partial\Sigma$ transversely and $\gamma_i \cap \partial\Sigma$ consists of the endpoints of γ_i. We may find disjoint discs D_i such that D_i intersects Σ at exactly γ_i and the boundary D_i, denoted by f_i is contained in $Y(K)$. This implies that the linking number of e_i and f_j is equal to zero if $i \neq j$ because f_j

is null-homologous in the complement of e_i. Moreover, f_i in $S^3 \backslash e_i$ is homologous to a meridional curve with one of the two orientations. Thus possibly after changing the orientation of f_j, we may assume that:

$$\mathrm{lk}(e_i, f_j) = \delta_{i,j}. \tag{5.4}$$

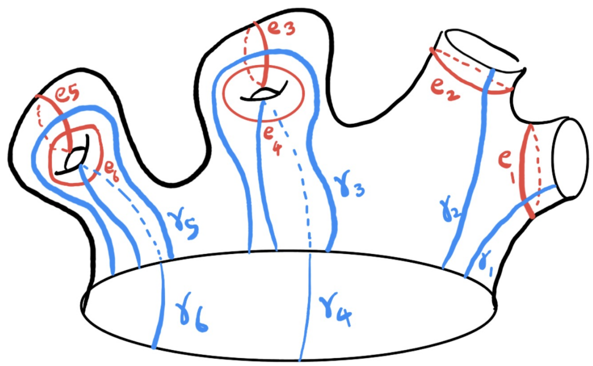


Figure 3: The standard basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ for $\Sigma_{2,3}$ and the corresponding embedded paths $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$

Now let $x \in H_1(\Sigma)$ be chosen such that $p(x, y) = 0$ for all $y \in H_1(Y(K))$. We can write x as a linear combination of the following form:

$$x = m_1 e_1 + \cdots + m_{2g+d-1}.$$

The identity $p(x, f_i) = 0$ implies that $m_i = 0$. In particular, $x = 0$.

Next, assume that there is $y \in H_1(Y(K))$ such that $p(x, y) = 0$ for all choices of x. There is a non-trivial linear combination of $f_1, \ldots, f_{2g+d-1}$ and y as follows such that:

$$z = n_1 f_1 + \cdots + n_{2g+d-1} f_{2g+d-1} + n_{2g+d} y = 0$$

Otherwise, we have a subgroup of $H_1(Y(K))$ which is isomorphic to $\mathbf{Z}^{2g+d}$, which is a contradiction because $H_1(Y(K)) = \mathbf{Z}^{2g+d-1}$. (See Subsection 5.1.) By taking the pairing of e_i and z, we can show that $n_i = 0$ for $1 \leqslant i \leqslant 2g + d - 1$. Thus we have $n_{2g+d} y = 0$. Since $H_1(Y(K))$ does not have any torsion element, $y = 0$. This verifies our claim. $\square$

Definition 5.5. The *Seifert form* for the Seifert surface Σ of K is defined to be the pairing:

$$q : H_1(\Sigma) \times H_1(\Sigma) \to \mathbf{Z}$$

defined as:

$$q(\alpha, \beta) := p(\alpha, \phi_*^+(\beta)).$$

Remark 5.6. There is a homeomorphism of the knot exterior $X(K)$ which maps α to $\phi_-(\alpha)$ and $\phi_*^+(\beta)$ to β. In particular, $q(\alpha, \beta)$ may be defined as the linking number of $\phi_*^-(\alpha)$ and β.

Example 5.7. Suppose Σ is the Seifert surface that is obtained by applying Seifert algorithm to the diagram of trefoil given by Figure 4. Then the closed loops f and β form a basis for $H_1(\Sigma)$. Moreover, we have:

$$q(\alpha,\alpha) = -1, \qquad q(\alpha,\beta) = 1, \qquad q(\beta,\alpha) = 0, \qquad q(\beta,\beta) = -1.$$

In particular, the intersection pairing is represented by the following matrix:

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

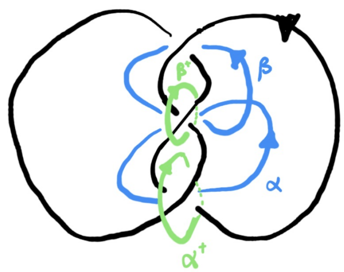


Figure 4: If Σ is the Seifert surface associated to the above digram, then α, β form a basis for $H_1(\Sigma)$. The homology classes $\phi^+(\alpha)$ and $\phi^+(\beta)$ are represented by the oriented loops α^+ and β^+.

The non-degeneracy of the pairing p implies that the maps $\phi^+_* : H_1(\Sigma) \to H_1(Y(K))$ and $\phi^-_* : H_1(\Sigma) \to H_1(Y(K))$ can be described in terms of the Seifert form. We fix a basis $e_1, \dots, e_{2g+d-1}$ for $H_1(\Sigma)$ and analogous to the proof of 5.3 we fix a basis $f_1, \dots, f_{2g+d-1}$ for $H_1(Y(K))$ such that:

$$p(e_i, f_j) = \delta_{i,j}.$$

Then the definition of the Seifert form implies that:

$$\phi^+_*(e_j) = \sum_i Q^i_j f_i. \tag{5.8}$$

where $Q^i_j := q(e_i, e_j)$. Remark 5.6 implies that we also have:

$$\phi^-_*(e_j) = \sum_i Q^j_i f_i. \tag{5.9}$$

Proposition 5.10. *A presentation matrix for the the* $\mathbf{Z}[t^{-1}, t]$*-module* $H_1(X_\infty(K))$ *is given by* $-Q + tQ^t$ *where* $Q = (Q^i_j)^{1\leqslant i\leqslant 2g+d-1}_{1\leqslant j\leqslant 2g+d-1}$.

Proof. Suppose U, V be subspaces of $X_\infty(K)$ given as follows:

$$U = \bigcup_i Y_{2i}(K) \qquad V = \bigcup_i Y_{2i+1}(K).$$

In particular, $U \cap V$ is the subspace $\bigcup_i \Sigma_i \subset X_\infty(K)$. Then Mayer-Vietoris sequence implies that:

$$H_1(U \cap V) \xrightarrow{g_1} H_1(U) \oplus H_1(V) \xrightarrow{h_1} H_1(X_\infty(K)) \xrightarrow{d_1} H_0(U \cap V) \xrightarrow{g_0} H_0(U) \oplus H_0(V). \quad (5.11)$$

Recall that the map g_i and h_i are given as follows:

$$g_i(\alpha) = (-\mathfrak{k}_*(\alpha), \mathfrak{l}_*(\alpha)) \qquad h_i(\beta, \sigma) = \mathfrak{i}_*(\beta) + \mathfrak{j}_*(\sigma)$$

where $\mathfrak{i} : U \to X$, $\mathfrak{j} : U \to X$, $\mathfrak{k} : U \cap V \to U$ and $\mathfrak{l} : U \cap V \to V$ are inclusion maps.

All the terms in (5.11) can be equipped with a $\mathbf{Z}[t^{-1}, t]$-module structure such that the morphisms are all module homomorphisms. We already defined the module structure on $H_j(X_\infty(K))$ and we only need to explain how t acts on:

$$(\dots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \dots) \in H_j(U \cap V) \quad (5.12)$$

with $\alpha_i \in H_j(\Sigma_i)$and

$$((\dots, \beta_{2i-2}, \beta_{2i}, \beta_{2i+2}, \dots), (\dots, \gamma_{2i-1}, \gamma_{2i+1}, \gamma_{2i+3}, \dots)) \in H_j(U) \oplus H_j(V) \quad (5.13)$$

with $\beta_{2i} \in H_j(Y_{2i}(K))$ and $\gamma_{2i+1} \in H_j(Y_{2i+1}(K))$. Then t acts on (5.12) by shifting to the right:

$$t \cdot (\dots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \dots) = (\dots, \alpha_{i-2}, \alpha_{i-1}, \alpha_i, \dots)$$

and it acts on (5.13)

$$t \cdot ((\dots, \beta_{2i-2}, \beta_{2i}, \beta_{2i+2}, \dots), (\dots, \gamma_{2i-1}, \gamma_{2i+1}, \gamma_{2i+3}, \dots)) =$$
$$= ((\dots, \gamma_{2i-3}, \gamma_{2i-1}, \gamma_{2i+1}, \dots), (\dots, \beta_{2i-2}, \beta_{2i}, \beta_{2i+2}, \dots))$$

It is easy to see that the maps g_i and h_i are $\mathbf{Z}[t^{-1}, t]$-module homomorphisms. We didn't give the definition of the maps d_i in the class. Once you know the definition, it is not hard to show that this map also respects the $\mathbf{Z}[t^{-1}, t]$-module structures.

Since Σ and $Y(K)$ are connected, elements of $H_0(U)$, $H_0(V)$ and $H_0(U \cap V)$ can be identified with elements of $\mathbf{Z}[t^{-1}, t]$. The map g_0 sends an element $p(t) \in H_0(U \cap V)$ to $(-p(t), p(t))$. In particular, g_0 is injective. Therefore, $H_1(X_\infty(K))$, as a $\mathbf{Z}[t^{-1}, t]$-module, can be identified with $H_1(U) \oplus H_1(V)$ modulo the image of g_1. Notice that $H_1(U \cap V)$ and $H_1(U) \oplus H_1(V)$ are both free $\mathbf{Z}[t^{-1}, t]$-modules with basis given by a $\mathbf{Z}$-basis of $H_1(\Sigma)$ and $H_1(Y(K))$. In particular, we can take the basis $e_1, \dots, e_{2g+d-1}$ for $H_1(\Sigma_0) \subset H_1(U \cap V)$ and the basis $f_1, \dots, f_{2g+d-1}$ for $H_1(Y_0(K)) \subset H_1(U) \oplus H_1(V)$. The map g_i sends the basis element e_i to $-\phi_*^+(e_i) + t\phi_*^-(e_i)$. Using (5.8) and (5.9), we can conclude that:

$$g_1(e_i) = \sum_j (-Q_j^i + tQ_i^j) f_j.$$

Thus $-Q + tQ^t$ is a presentation matrix for the module $H_1(X_\infty(K))$.

□

Corollary 5.14. *The first elementary ideal of the Alexander module is principal and is generated by* $\det(-Q + tQ^t)$*. In particular, the Alexander polynomial of* K *is equal to* $\det(-Q + tQ^t)$ *(up to multiplication by* $\pm t^n$*).*

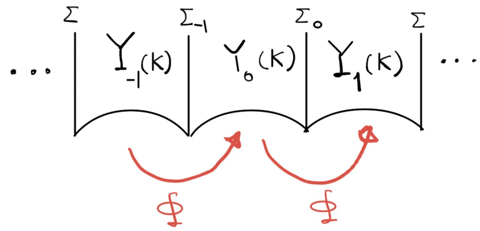


Figure 5

Example 5.15. Proposition 5.10 implies that a presentation matrix for $H_1(X_\infty(T_{2,3}))$ is given as:

$$\begin{bmatrix} 1-t & -1 \\ t & 1-t \end{bmatrix}.$$

Therefore, the Alexander polynomial of trefoil is equal to $t^2 - t + 1$.

References

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978 ↑4