Lecture 17

1 Introduction

In this lecture, we discuss some of the basic properties of Alexander polynomials of knots. Recall that so far Alexander polynomial of an oriented link K, denoted by $\Delta_K(t)$, is defined up to multiplication by $\pm t^n$. So it is convenient to introduce a notation for this possibility. If $p_1(t), p_2(t)$ are elements of $\mathbf{Z}[t^{-1}, t]$ such that $p_2(t) = \pm t^n p_1(t)$, then we write $p_1(t) \cong p_2(t)$.

2 Basic Properties of Alexander Polynomial

Proposition 2.1. *For any oriented link K, we have $\Delta_K(t^{-1}) \cong \Delta_K(t)$.*

Proof. Suppose Q is the matrix of a Seifert form of K, which has size m. The above identity is an immediate consequence of the characterization of $\Delta_K(t)$ in terms of Q:

$$\Delta_K(t^{-1}) = \det(-Q + t^{-1}Q^t) = (-t)^{-m} \det(tQ - Q^t) \cong \det(tQ - Q^t) = \det(tQ^t - Q) = \Delta_K(t).$$

□

Proposition 2.2. *For any oriented knot K, we have $\Delta_K(1) = \pm 1$. For any oriented link K with more than two components, we have $\Delta_K(1) = 0$.*

In order to prove the above lemma, we firstly consider the following elementary lemma about linking numbers:

Lemma 2.3. *Suppose A is an embedding $\phi : [-1, 1] \times S^1 \to S^3$ of a cylinder into S^3. Suppose γ_t denotes the knot given by $\phi|_{\{t\}\times S^1}$. Then we have:*

(i) If K is a knot in S^3 which is disjoint from A, then $\mathrm{lk}(K, \gamma_1) = \mathrm{lk}(K, \gamma_{-1})$.

(ii) If K is a knot in S^3 which intersects A transversely in one point, then $\mathrm{lk}(K, \gamma_1) = \mathrm{lk}(K, \gamma_{-1}) \pm 1$.

(iii) $\mathrm{lk}(\gamma_0, \gamma_1) = \mathrm{lk}(\gamma_0, \gamma_{-1})$.

Proof. In (i), the cylinder A allows us to show that γ_1 and γ_{-1} represent the same homology class in $S^3\backslash K$. In (ii), the cylinder A minus a small neighborhood of the intersection point $A \cap K$ shows that the difference of homology classes of γ_1 and γ_{-1} is homologuos to $\pm\mu$ where μ is the homology class of a meridian of K. Finally, in (iii), we can push γ_0 off A by a small perturbation to obtain γ_0' which represent the same homology class as γ_0 in $S^3\backslash(\gamma_1 \cup \gamma_{-1})$. Therefore, we have:

$$\mathrm{lk}(\gamma_0, \gamma_1) - \mathrm{lk}(\gamma_0, \gamma_{-1}) = \mathrm{lk}(\gamma_0', \gamma_1) - \mathrm{lk}(\gamma_0', \gamma_{-1}) = 0.$$

For the second identity, we use part (i). □

Proof of Proposition 2.2. Suppose K has d connected components and Σ is a Seifert surface for K which is diffeomorphic to $\Sigma_{g,d}$. Then $H_1(\Sigma)$ has a basis of the form $e_1, e_2, \ldots, e_{2g+d-1}$ such that for e_{2i-1} and e_{2i} intersect transversely in exactly one point for $1 \leqslant i \leqslant g$, and all other pairs of e_i and e_j are disjoint from each other. Then we have $\Delta_K(1) = \det(Q^t - Q)$. The (k, l)-entry of $Q^t - Q$, denoted by a_l^k is given as follows:

$$\mathrm{lk}(e_k, \phi_*^-(e_l)) - \mathrm{lk}(e_k, \phi_*^+(e_l)) = \mathrm{lk}(e_l, \phi_*^+(e_k)) - \mathrm{lk}(e_l, \phi_*^-(e_k)).$$

Lemma 2.3 the only non-vanishing possible values of a_l^k are given as follows:

$$a_{2i}^{2i-1} = -a_{2i-1}^{2i} = \pm 1.$$

In particular, $Q^t - Q$ is the matrix which has g diagonal 2×2 blocks of the following form:

$$\pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the remaining entries are zero. Thus $\det(Q^t - Q)$ is ± 1 if $d = 1$ and is 0 if $d > 1$. This completes the proof. □

We can use the previous two propositions to remove the ambiguity in the definition of $\Delta_K(t)$ in the case that K is a knot. Proposition 2.1 implies that for any oriented link K, if we have:

$$\Delta_K(t) \cong a_0 + a_1 t + \cdots + a_N t^N$$

with $a_0,\ a_N \neq 0$, then $a_{N-k} = \pm a_k$ for any $0 \leqslant k \leqslant N$. If K is a knot, then N has to be an even integer $2M$, otherwise $\Delta_K(1)$ is even, which is a contradiction. Moreover, the sign in $a_{2M-k} = \pm a_k$ has to be positive, otherwise the term $a_M = 0$, which implies again the contradictory result that $\Delta_K(1)$ is an even integer. In summary, after relabeling the indices and changing the representative, we can assume that:

$$\Delta_K(t) \cong b_0 + b_1(t + t^{-1}) \cdots + b_M(t^M + t^{-M})$$

where b_0 is an odd integer. There is still a sign ambiguity and multiplying the above polynomial by -1 gives another representative with the similar from. We may avoid this ambiguity by requiring that $\Delta_K(1) = 1$.

Proposition 2.4. *Alexander polynomial $\Delta_K(t)$ of an oriented link K satisfies the following properties with respect to basic operations on knots:*

(i) If rK denotes K with the reverse orientation, then $\Delta_{rK}(t) \cong \Delta_K(t)$.

(ii) If $\overline{K}$ denotes the reflection of K, then $\Delta_{\overline{K}}(t) \cong \Delta_K(t)$.

(iii) $\Delta_{K_1 \# K_2}(t) \cong \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)$.

Proof. Suppose Σ is a Seifert surface for K. Then Σ with the reverse orientation gives a Seifert matrix for rK, and reflection of Σ (with the induced orientation by reflection) gives a Seifert surface for $\overline{\Sigma}$. From this it is easy to see that if Q is the Seifert form matrix associated to Σ, then Q^t and $-Q$ give Seifert matrices for rK and $\overline{K}$.

Next, let Σ_1 and Σ_2 be Seifert surfaces for K_1 and K_2. Then the boundary sum $\Sigma_1 \natural \Sigma_2$ is a Seifert surface for $K_1 \# K_2$. If Q_1 and Q_2 are the Seifert form matrices associated to Σ_1 and Σ_2, then the matrix with two diagonal blocks Q_1 and Q_2 is the Seifert matrix associated to $\Sigma_1 \natural \Sigma_2$. $\square$

Recall that for $p(t) \in \mathbf{Z}[t^{-1}, t]$, we defined $M(p(t))$ (resp. $m(p(t))$) to be the degree of the largest (resp. smallest) power of t in $p(t)$.

Proposition 2.5. *If Σ is a Seifert surface of genus g for a link K, then $M(\Delta_k(t)) - m(\Delta_k(t)) \leqslant 2g + d - 1$. In particular, for any knot K:*

$$g(K) \geqslant \frac{M(\Delta_k(t)) - m(\Delta_k(t))}{2}.$$

Notice that the expression on the left does not change if we change the representative of $\Delta_K(t)$ by multiplying with an expression of the form $\pm t^n$.

Proof. If Σ is homeomorphic to $\Sigma_{g,d}$, then the associated Seifert form matrix Q has size $2g + d - 1$. Therefore, the largest power of $\det(-Q + tQ^t)$ is at most $2g + d - 1$ and the smallest power of $\det(-Q + tQ^t)$ is at least 0. This proves the desired result. $\square$

Proposition 2.6. *If $L = K_1 \cup K_2$ is a link with connected components such that there are disjoint Seifert surfaces Σ_1 and Σ_2 for K_1 and K_2, then $\Delta_L(t) = 0$.*

Proof. We fix bases $e_1, \ldots, e_{2g_1}$ for $H_1(\Sigma_1)$ and $f_1, \ldots, f_{2g_2}$ for $H_1(\Sigma_2)$. We also take the connect Σ_1 and Σ_2 by a thin tube to form a Seifert surface Σ for L. Let h be a small closed loop which goes around the tube once. Then $e_1, e_2, \ldots, e_{2g_1}, f_1, f_2, \ldots, f_{2g_2}$ and h form a basis for $H_1(\Sigma)$. Moreover, the Seifert pairing of q with any other basis element vanishes. This shows that the last column and the last row of the Seifert matrix has vanishing entries. $\square$