Lectures 18 and 19

1 Introduction

In the previous lectures, we saw that how we can use a Seifert surface Σ of an oriented link K to define a square matrix which is called the Seifert matrix. We also stated a theorem that explains how Alexander polynomial can be computed out of the Seifert matrix. The Seifert matrix clearly depends on the choice of the Seifert surface. For example, even the size of the Seifert matrix changes as we change the topology of Σ. In this lecture, we closely study the effects of changing the Seifert surface on the Seifert matrix. This allows us the ambiguity in the definition of Alexander polynomial of K.[1] This would in turn lead to a new characterization of Alexander polynomial in terms of link diagrams.

Definition 1.1. Let A be an $n \times n$ matrix with integer entries. An *elementary enlargement* of A is a matrix B which has the following form of one of the following matrices:

$$\left[\begin{array}{c|c|c} A & \zeta & 0 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline \eta^t & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right], \tag{1.2}$$

where $\zeta, \eta \in \mathbf{R}^n$. Under this assumption, we say A is an *elementary reduction* of B.

We say A is *unimodular congruent* ti B if there is a matrix P with integer entries and $\det(P) = \pm 1$ such that $A = P^t AP$.

Definition 1.3. We say A and B are S-equivalent, if they are related to each other by a sequence of elementary enlargements, elementary reductions and unimodular congruences.

Theorem 1.4. ([Lic97, Theorem 8.4]) *Any two Seifert matrices of an oriented link K are S-equivalent to each other.*

Let A be a Seifert matrix for a link K. Then we define the *Conway normalized Alexander polynomial* of K to be:

$$\Delta_K(t) := \det(-t^{\frac{1}{2}}A + t^{\frac{1}{2}}A^t) \in \mathbf{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]. \tag{1.5}$$

Notice that if m is the size of A, then we have $\Delta_K(t) = t^{-\frac{m}{2}}\det(A + tA^t)$. In particular, this agrees with the previous characterization of Alexander polynomial (which was already ambiguous up $\pm t^n$) up to a half integer power of t.

[1] We already know how to remove this ambiguity for a knot, but not for link with more than one connected component.

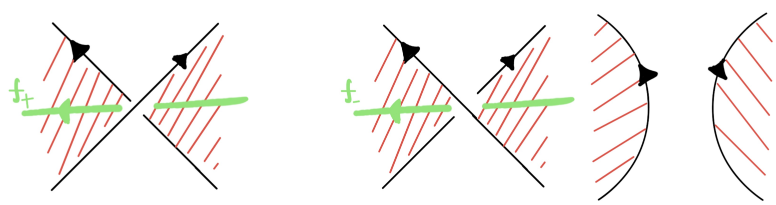
The following proposition is a consequence of Theorem 1.4 and can be proved by studying the behavior of (1.5) with respect to elementary enlargements, elementary reductions and unimodular congruences:

Corollary 1.6. ([Lic97, Theorem 8.5]) *Conway normalized Alexander polynomial is an invariant of K and does not depend on the choice of the Seifert matrix.*

Theorem 1.7. $\Delta_K(t) \in \mathbf{Z}[t^{-\frac{1}{2}}, t^{\frac{1}{2}}]$ *is uniquely characterized by the following properties:*

(i) $\Delta_U(t) = 1$ *where U is the unknot.*

(ii) $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta_{L_0}(t)$ *where L_+ is an arbitrary link with a given diagram and L_-, L_0 are obtained by changing L_+ in a neighborhood of a crossing as in Figure 1.*



(a) L_+ (b) L_- (c) L_0

Figure 1: Diagrams of the three oriented links L_+, L_- and L_0

Proof. Similar to our second characterization of Jones polynomial, there is at most one polynomial for each link K which satisfies the above two properties. To finish the proof of the above theorem, we need to show that the definition (1.5) satisfies the two properties. It is clear that $\Delta_U(t) = 1$. To see the other property, we apply Seifert algorithm to the given diagram of L_+ and the induced diagrams of L_- and L_0. (See Figure 1.) We denote these Seifert surfaces with Σ_+, Σ_- and Σ_0. The Euler characteristics of these surfaces are related as follows:

$$\chi(\Sigma_+) = \chi(\Sigma_-) = \chi(\Sigma_0) - 1.$$

In fact, if $\Sigma_0 \cong \Sigma_{g,d}$, then $\Sigma_+ \cong \Sigma_-$ is homeomorphic to $\Sigma_{g,d+1}$ or $\Sigma_{g+1,d-1}$. In any case, if we pick a basis of oriented closed loops for $H_1(\Sigma_0)$, then they induce a set of loops in Σ_+ (resp. Σ_-) which together with a loop f_+ (resp. f_-) as in Figure 1a (resp. 1b) gives a basis for $H_1(\Sigma_+)$ (resp. $H_1(\Sigma_-)$). The loops f_+ and f_- agree with each other outside the crossings of L_+ and L_- which are shown in the figure. In particular, if N is the self-pairing of f_+ with respect to the Seifert form of L_+ induced by Σ_+, then the self-pairing of f_- with respect to the Seifert form of L_- induced by Σ_- is equal to $N + 1$. The pairing of f_+ with the other basis elements are equal to the pairing of f_- with other basis elements, and the pairing of other basis elements with each other are equal to the one given by Σ_0. In particular, the Seifert matrices associated to Σ_+, Σ_- and Σ_0 have the following form:

$$A_+ = \left[\begin{array}{c|c} N & \zeta^t \\ \hline \eta & A \end{array}\right] \qquad A_- = \left[\begin{array}{c|c} N+1 & \zeta^t \\ \hline \eta & A \end{array}\right] \qquad A_0 = A \tag{1.8}$$

It is easy to see (ii) from these identities. □

Example 1.9. The Alexander polynomial of a the unknit with at least two components is equal to 0. For example in the case of the link with two components, this can be see by applying part (ii) of the previous theorem to the knot digram in 2. A similar argument can be used to prove the general case.

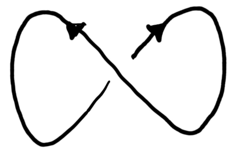


Figure 2

Corollary 1.10. *(i)* $\Delta_K(1) = 1$ *if* K *is a knot and* $\Delta_K(1) = 0$ *if* K *is a link with at least two connected components.*

(ii) $\Delta_K(t)$ *is a polynomial in* $y = t^{-\frac{1}{2}} - t^{\frac{1}{2}}$.

Proof. Theorem (1.7) implies that $\Delta_{L_+}(1) = \Delta_{L_-}(1)$. Since any link can be turn into an unlink with the same number of connected components by a number of crossing changes, the first claim follows from the evaluation of Alexander polynomials of unlinks. The second claim follows from induction similar to the proof of the analogous property for Jones polynomial. □

We define $\nabla_K(y) \in \mathbf{Z}[y]$ to the polynomial such that:

$$\nabla_K(t^{-\frac{1}{2}} - t^{\frac{1}{2}}) = t^{-\frac{1}{2}} - t^{\frac{1}{2}}.$$

Corollary 1.11. *If* $\nabla_K(y) = \sum_{i \geqslant 0} a_i(K) y^i$, *then we have:*

(i) $a_i(K) = 0$ *for* $i \equiv \#L \mod 2$ *or* $i < \#L - 1$.

(ii) If K *is a knot, then* $a_0(K) = 1$.

(iii) If K *is a link with two components, then* $a_1(K)$ *is equal to the linking number of the two components of* K.

Proof. The second part of Theorem 1.7 can be written as follows:

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z).$$

Moreover, we have:

$$\#L_+ = \#L_- = \#L_0 \pm 1.$$

Therefore, an inductive argument as before can be used to prove (i). To be more detailed, one uses induction on the crossing number and then induction on the following quantity for a link with crossing number n:

$\mathfrak{C}(K) = \min\{$number of crossing changes required to turn D into a diagram for an unlink $|$

D is a diagram of K with n crossings$\}$

Part (ii) is a consequence of Corollary 1.10. To see (iii) note that part (ii) and Theorem 1.7 imply that:

$$a_2(L_+) - a_2(L_-) = 1$$

where L_+ is a link with two components and L_- is obtained by turning one of the positive crossings between the two components of L_+ into a negative crossing. Now theorem follows from the diagramatic definition of linking number in Problem Set 3. □

References

[Lic97] W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. MR1472978 ↑1, 2