Lecture 7

1 Intorduction

For any L in S^3, we may form a 3-manifold $X(L)$ by taking the closure of the complement of a regular neighborhood of L. This 3-manifold, called the *exterior of* L has boundary which is a 2-dimensional torus. The interior of $X(L)$ is homeomorphic to the complement of L. It is clear from the definition of equivalence of links that if two links L_1 and L_2 are equivalent to each other, then the 3-manifolds $X(L_1)$ and $X(L_2)$ are homeomorphic to each other. In fact, the inverse of this observation holds for knots. Recall that we already saw examples of distinct links with homeomorphic exteriors (Problem Set 2).

Theorem 1.1. *If two knots K_1 and K_2 have homeomorphic exteriors, then there are equivalent to each other.*

Since the homeomorphism type of $X(L)$ is an invariant of L, we may apply constructions from algebraic topology to construct algebraic invariants of L. For example, we may consider fundamental groups, higher homotopy groups or homology groups of $X(L)$ to associate various groups to $X(L)$, which are invariant with respect to equivalence of knots. Among these invariants, the only interesting one is the fundamental group of $X(L)$. In fact, this invariant turns out to be very strong. For example, we have the following theorem:

Theorem 1.2. *If K_1 and K_2 are two prime knots such that the fundamental groups of $X(K_1)$ and $X(K_2)$ are isomorphic to each other, then K_1 and K_2 are equivalent to each other.*

In this lecture, we firstly review the definition of fundamental groups of topological spaces. Next, we state some of the basic properties of fundamental groups.

2 Fundamental Groups

Suppose X is a topological space and $x_0 \in X$ is a base point of X. A *loop* based at x_0 is a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1) = x_0$. Suppose γ_0 and γ_1 are two loops based at x_0. We say γ_0 is homotopic to γ_1 among all based loops if there is a continuous map $H : [0,1] \times [0,1] \to X$ such that:

(i) For $i = 0, 1$ and any $t \in [0,1]$, we have $H(i,t) = \gamma_i(t)$;

(ii) For $j = 0, 1$ and any $s \in [0, 1]$, we have $H(s, j) = x_0$.

The above notion of homotopy defines an equivalence relation among all based loops in X. The fundamental group of X, denoted by $\pi_1(X, x_0)$, is given by the equivalence classes of this equivalence relation.

As its name suggests, we can equip the fundamental group $\pi_1(X, x_0)$ with a group structure. If $\gamma_0, \gamma_1 : [0, 1] \to X$ are two paths based at x, we define the composed loop $\gamma_0 * \gamma_1 : [0, 1] \to X$ to be the loop based at x which is defined as follows:

$$\gamma_0 * \gamma_1(t) = \begin{cases} \gamma_0(2t) & 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

Definition of composition of loops is compatible with respect to the notion of homotopy. That is to say, if the based loops γ_0 and γ_0' are homotopic to each other and γ_1 and γ_1' are two other homotopic based loops, then $\gamma_0 * \gamma_1$ and $\gamma_0' * \gamma_1'$ are homotopic to each other among loops based at x_0. Therefore, the composition of loops induces a notion of multiplication on the elements of $\pi_1(X, x_0)$. It is straightforward to show that this multiplication map is associative, admits an identity element and any element of $\pi_1(X, x_0)$ has a multiplicative inverse.

These are some of the basic properties of fundamental groups:

1. If x_0 and x_1 belong to the same connected components of X, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic to each other.

2. Let $f : X \to Y$ be a continuous map which sends $x_0 \in X$ to $y_0 \in Y$. Then f induces a group homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ which maps a loop γ in X based at x_0 to $f \circ \gamma$. If f is a homeomorphism, then f_* is an isomorphism of fundamental groups. More generally, the same claim holds if f is a homotopy equivalence. Recall that f is a homotopy equivalence if there a continuous map $g : Y \to X$ such that $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are both homotopic to the identity maps.

3. If X and Y are two topological spaces with base points x_0 and y_0, then $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

4. If X is a topological space that consists of only one element x_0, then $\pi_1(X, x_0) = \{0\}$. More generally, if X is a contractible topological space, then $\pi_1(X, x_0) = \{0\}$. This is true because the constant map from X to the topological space with one element is a homotopy equivalence.

5. Identify S^1 with the quotient space $\mathbf{R}/\mathbf{Z}$ and let 0 be the base point of S^1. Then $\pi_1(S^1, 0) \cong \mathbf{Z}$. A generator of this topological space is given by the based loop $\gamma(t) = [t]$.

Usually we drop x_0 from our nation $\pi_1(X, x_0)$ for the fundamental groups. This is partly justified because of the first property: if X is a path connected space, then the isomorphism class of $\pi_1(X, x_0)$ is independent of the choice of the base point x_0.

In our class, we are mainly interested in $\pi_1(X(K))$ where K is a knot in S^3. In this case, the fundamental group is called the *knot group* of K. Of course, we can similarly consider fundamental groups of link exteriors.

Example 2.1. The exterior of the unknot U is homeomorphic to $S^1 \times D^2$ where D^2 is the 2-dimensional disc. Since D^2 is a contractible space, the knot group $\pi_1(X(U))$ is isomorphic to $\mathbf{Z}$.

Example 2.2. In an exercise in Problem Set 1, you showed that the exterior of the Hopf link H is homeomorphic to $S^1 \times S^1 \times [0, 1]$. Thus $\pi_1(X(H))$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.