Lecture 8

1 Intorduction

In this lecture, we firstly state Seifert-Van Kampen Theorem, which is a very useful theorem for computing fundamental groups of topological spaces. The main idea of this theorem is to decompose a topological space into simpler subspaces and describe the fundamental group of our topological space in terms of the invariants of the fundamental groups of its subspaces. Next, we state Wirtinger presentation which gives us an algorithm to write a presentation of a link group in terms of a diagram for the link.

2 Seifert-Van Kampen Theorem

Theorem 2.1. *Suppose X is the union of two path connected open subspaces U and V such that $U \cap V$ is also path connected. We choose a point $x_0 \in U \cap V$ and use it to define base points for the topological subspaces X, U, V and $U \cap V$. Suppose $i_* : \pi_1(U) \to \pi_1(X)$ and $j_* : \pi_1(V) \to \pi_1(X)$ are given by inclusion maps. Let $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)$ be the maps induced by i_* and j_* where $\pi_1(U) * \pi_1(V)$ denotes the free product of groups $\pi_1(U)$ and $\pi_1(V)$. Then Φ is a surjective map. Moreover, the kernel of this map is the smallest normal subgroup of $\pi_1(U) * \pi_1(V)$ which has the elements of the form $k_*(\gamma) * l_*(\gamma)^{-1}$ where $k : U \cap V \to U$ and $l : V \cap V \to U$ are the inclusion maps.*

Example 2.2. Let B_n be the open unit disc in the plane where n points are removed. Then we claim that the fundamental group of B_n is a free group with n generators. That is to say, it is isomorphic to $\underbrace{\mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z}}_{n}$. In the case that $n = 1$, B_1 is homeomorphic to $(0, 1) \times S^1$. Therefore, $\pi_1(B_1) = \mathbf{Z}$. For general n, we may decompose B_n as the union of two open sets U and V where U is homeomorphic to B_{n-1}, V is homeomorphic to B_1 and $U \cap V$ is homeomorphic to an open disc. Therefore, the claim follows from induction on n. A closely related space to B_n is X_n, the space given as the wedge of n copies of the circle S^1. It is easy to see that there is a deformation retraction from B_n to X_n.

Example 2.3. In a homework problem, you shall show that the torus knot $T_{p,q}$ has the same homotopy type as the quotient $X_{p,q}$ of the cylinder $S^1 \times [0, 1]$ with respect to the following action [Hat02, Example 1.24]:

$$(\theta + \frac{1}{p}, 0) \sim (\theta, 0) \qquad (\theta + \frac{1}{q}, 1) \sim (\theta, 1).$$

Then $X_{p,q}$ can be written as the union $U \cup V$ where U is the quotient of $S^1 \times [0, \frac{2}{3})$ with respect to the following action:

$$(\theta + \frac{1}{p}, 0) \sim (\theta, 0),$$

and V is the quotient of $S^1 \times (\frac{1}{3}, 1]$ with respect to the following action:

$$(\theta + \frac{1}{q}, 1) \sim (\theta, 1).$$

The intersection $U \cap V$ is equal to $S^1 \times (\frac{1}{3}, \frac{2}{3})$. Since U and V both retracts to S^1, the fundamental groups of U, V and $U \cap V$ are all diffeomorphic to $\mathbf{Z}$. We denote the generators of $\pi_1(U)$ and $\pi_1(V)$ with x and y. The inclusion of $U \cap V$ in U (resp. V) maps the generator of $\pi_1(U \cap V)$ to x^p (resp. y^q). Seifert-Van Kampen theorem implies that $\pi_1(X_{p,q})$ (and hence the knot group of $T_{p,q}$) is freely generated by x and y with the relation that $x^p = y^q$:

$$\pi_1(X(T_{p,q})) = \langle x, y \mid x^p = y^q \rangle.$$

Example 2.4. Suppose Σ_g is the orientable surface of genus g. One way of constructing Σ_g is given by taking a $4g$-gon as in Figure 1, pair the edges as in the figure and then identify the edges which have the same labels. To visualize why this is a genus g Riemann surface, notice that all vertices are of this polygon are identified with each other. The resulting space is a 2-sphere with $2g$ holes in it, where the boundary of these holes are matched in pairs. Identifying these boundary circles in pairs is the same as adding g handles to the 2-sphere with $2g$ holes. Suppose Σ_g° denote the Riemann surface of genus g

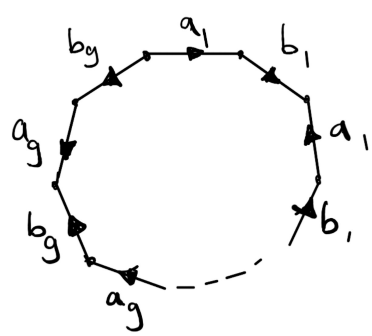


Figure 1

with one boundary component. That is to say, we remove an open disc from Σ_g to obtain Σ_g°. From the above description of Σ_g it is clear that Σ_g° has the same homotopy type as X_{2g}, the wedge of $2g$ circles. Therefore, we have:

$$\pi_1(\Sigma_g^\circ) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \rangle \tag{2.5}$$

Now an application of Seifert-Van Kampen theorem implies that:

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle \tag{2.6}$$

3 Wirtinger Presentation

If D is a diagram for a link L, Seifert-van Kampen theorem can be used to give an algorithm to give a presentation for the fundamental group of the exterior of L, which is called the *Wirtinger presentation*. Suppose your diagrams has m segments denoted by $\alpha_1, \alpha_2, \ldots, \alpha_m$. For each segment α_i, we introduce an element g_i of $\pi_1(X(L))$ as follows. Firstly pick a base point x_0 on top of the diagram. (Imagine that you are looking at the diagram from the top and the tip of your nose is the base point.) Take also a small arrow w_i passing under α_i as in Figure 2. Take a straight path from the base point to the tail of w_i, then travel along w_i and finally come back to the base point by a straight path from the head of w_i.

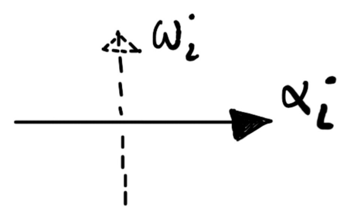


Figure 2

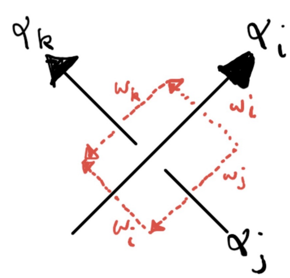
Given any crossing c of the diagram D, we can define a relations among the elements g_i as follows. If three segments α_i, α_j and α_k meet each other in a crossing as in Figure 3a, then the relation is $g_k = g_i^{-1} g_j g_i$, and if they meet as in Figure 3b, then the relation is $g_k = g_i g_j g_i^{-1}$. It is not hard to convince yourself that these are in fact relations in the fundamental group of $X(L)$. (Exercise: convince yourself!) If our diagram D has crossings $c_1, \ldots, c_n$, then we obtain relations $r_1, \ldots, r_n$.

Theorem 3.1. *The group $\pi_1(X(L))$ can be presented as follows:*

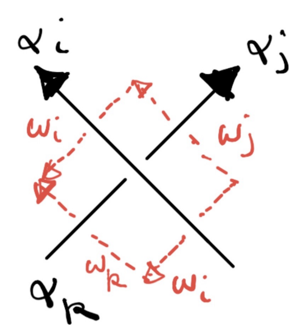
$$\langle g_1, g_2, \ldots, g_m \mid r_1, r_2 \ldots, r_n \rangle.$$

In fact, we can drop any of the relations r_i from the above presentation.

For a proof, See [?Rolfsen, Section 3.D.].



(a)



(b)

Figure 3

Example 3.2. Wirtinger presentation immediately implies that the link group of the unlink U_n is a free group generated by n elements.

Example 3.3. We may use Wirtinger presentation to give another description of the knot group of the trefoil. Suppose x, y and z are the elements of the knot group of the trefoil associated to the three arcs of the diagram in Figure 4. Then the relations associated to the crossings of the diagram imply that

$$yx = zy = xz$$

Therefore, we can use these relations to solve for z. Moreover, we obtain a relation in terms of x and y which is $xyx = yxy$. Therefore, we obtain the following presentations for the knot group of the trefoil:

$$\pi_1(X(T_{2,3})) = \langle x, y \mid xyx = yxy \rangle.$$

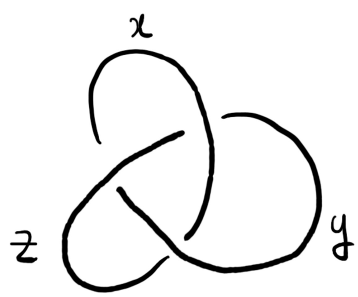


Figure 4

References

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354 ↑1