Lecture 9

1 Introduction

Fundamental groups of topological spaces tend to be non-abelian groups. This makes it often hard to get a good grasp on fundamental groups. One way to simplify the picture is to consider an *abelianized* version of fundamental groups. This goal can be achieved by introducing the first homology group of a topological space. In fact, one can easily generalize the definition of first homology group to define homology groups of arbitrary degree. In this lecture, we give the definition of the first homology group of a topological space and sketch how it can be generalized to give homology groups of arbitrary degree.

2 Homology Groups

Let X be a topological space. Motivated by the definition of the fundamental group of X, let $C_1(X)$ denote the free abelian group generated by paths in X. To be more detailed, an element of $C_1(X)$ consists of all formal linear combinations of the following form:

$$\alpha = m_1 \cdot \gamma_1 + m_2 \cdot \gamma_2 + \cdots + m_k \cdot \gamma_k \tag{2.1}$$

where m_i is an integer and γ_i is a continuous map from $[0, 1]$ to X. The addition of two such paths is defined in the obvious way.

We can form the variation $C_i(X)$ of $C_1(X)$ by replacing paths with i*-dimensional paths* in X. We start with the simplest case that $i = 0$. The abelian group $C_0(X)$ consists of elements of the following form:

$$\beta = m_1 \cdot x_1 + m_2 \cdot x_2 + \cdots + m_k \cdot x_k \tag{2.2}$$

where m_i is an integer and x_i is a point in X. There is a homomorphism $\partial_1 : C_1 \to C_0$. For example, $\partial_1(\alpha)$ for the element α is equal to:

$$\partial_1(\alpha) := m_1(\gamma_1(1) - \gamma_1(0)) + m_2(\gamma_2(1) - \gamma_2(0)) + \cdots + m_k(\gamma_k(1) - \gamma_k(0))$$

Notice that starting with any element in the kernel of ∂_1, we can from a union of closed loops in X. Moreover, $\text{coker}(\partial_1) := C_0(X)/\text{im}(\partial_1)$ can be identified with $\mathbf{Z}^n$, a free abelian group of rank n, where n is the path connected components of X. This co-kernel is the 0-th homology group of X, denoted by $H_0(X)$.

We may form a sub-group $\widetilde{C}_0$ of C_0 consisting of elements as in (2.2) where $\sum_i m_i = 0$. Clearly, the image of ∂_1 is in $\widetilde{C}_0$. Therefore, we can form the quotient $\widetilde{H}_0(X) := \widetilde{C}_0/\operatorname{im}(\partial_1)$ which is called the 0-th reduced homology group of X and is denoted by $\mathbf{Z}^{n-1}$, where n is again the number of the connected components of X. Sometimes it is more convenient to work with alternative version of the 0-th homology group.

Next, we define $C_2(X)$. An element of $C_2(X)$ is a formal linear combination of the following from:

$$\tau = m_1 \cdot \sigma_1 + m_2 \cdot \sigma_2 + \cdots + m_k \cdot \sigma_k \tag{2.3}$$

where σ_i is a map from the 2-dimensional simplex Δ (Figure **??**) to X. We also fix identifications of the edges of Δ with the interval $[0, 1]$. This allows us to define a map $\partial_2 : C_2(X) \to C_1(X)$ whose value at the element in (2.3) is equal to:

$$\partial_2(\tau) = m_1 \cdot (\sigma_1(e_1) + \sigma_1(e_2) - \sigma_1(e_3)) + \cdots + m_k \cdot (\sigma_1(e_1) + \sigma_1(e_2) - \sigma_1(e_3))$$

The following lemma is an immediate consequence of definitions of ∂_1 and ∂_2:

Lemma 2.4. $\partial_1 \circ \partial_2 = 0$*, i.e.,* $\operatorname{im}(\partial_2) \subset \ker(\partial_1)$

Definition 2.5. The first homology of X, denoted by $H_1(X)$, is defined to be the quotient $\ker(\partial_1)/\operatorname{im}(\partial_2)$.

In general, we can define $C_i(X)$ by considering maps from the i-dimensional simplex to X. Restriction of maps to boundaries of i-dimensional simplices produces a map $\partial_i : C_i(X) \to C_{i-1}(X)$ with the property that $\partial_{i-1} \circ \partial_i$. Therefore, by imitating Definition 2.5, we define the homology group $H_i(X)$ to be $\ker(\partial_i)/\operatorname{im}(\partial_{\mathrm{i}+1})$. In our class we will be mainly interested in $H_1(X)$ for various choices of topological spaces X which are constructed out knots and links, e.g., the knot complement.

Remark 2.6. Let X be a path connected topological space. Any element of $\pi_1(X)$ can be represented by a based loop $\gamma : S^1 \to X$. Any such element represents the trivial element if there is a map $\Gamma : D^2 \to X$ such that $\Gamma|_{\partial D^2} = \gamma$. Here D^2 is the 2-dimensional disc. It is not hard to see that any element of $H_1(X)$ can be also represented by a closed loop $\gamma : S^1 \to X$. One difference in the homology case is that the loop γ is not required to be based at a base point. The other difference is that a loop γ represents the trivial element in $H_1(X)$ if for some g there is map Γ from Σ_g°, the Riemann surface of genus g with one boundary component, to X such that the restriction of Γ to the boundary of Σ_g° is equal to γ.

3 Some Basic Properties of Homology Groups

Analogous to fundamental groups, homology groups are functorial. This means that any continuous map $f : X \to Y$ of topological spaces induces a homomorphism $f_* : H_i(X) \to H_i(Y)$. For example, if $\gamma : [0, 1] \to X$ gives an element of $C_1(X)$, then $f_*(\gamma) := f \circ \gamma : [0, 1] \to X$ gives an element of $C_1(Y)$. Similarly, we can define maps $f_* : C_i(X) \to C_i(Y)$. These maps satisfy the identity $f_* \circ \partial_i = \partial_i \circ f_*$. Thus we have an induced map $f_* : H_i(X) \to H_i(Y)$. As in the case of fundamental groups, if f is a homeomorphism, or more generally a homotopy equivalence, then f_* is an isomorphism.

A useful tool for computing homology groups is the *Mayer-Vietoris sequence*. Suppose X is a topological space, U and V are two open subspaces of X such that $X = U \cup V$. The Mayer-Vietoris sequence is analogous to the Seifert-Van Kampen theorem: it allows us to obtain information about the homology groups of X in terms of homology groups of U, V and $U \cap V$. Firstly recall that a sequence of abelian groups and homomorphisms as below is a chain complex if $f_{j+1} \circ f_j = 0$ for all choices of j:

$$\dots \xrightarrow{f_{i+2}} C_{i+1} \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} C_{i-1} \xrightarrow{f_{i-1}} C_{i-2} \xrightarrow{f_{i-2}} \dots . \tag{3.1}$$

This sequence is *exact* if $\ker(f_i) = \operatorname{im}(f_{i+1})$.

The Mayer-Vietoris sequence states that the homology groups of the spaces X, U, V and $U \cap V$ fit into an exact sequence of the following form:

$$\dots \xrightarrow{d_{i+1}} H_i(U \cap V) \xrightarrow{g_i} H_i(U) \oplus H_i(V) \xrightarrow{h_i} H_i(X) \xrightarrow{d_i} H_{i-1}(U \cap V) \xrightarrow{g_{i-1}} \dots$$
$$\dots \xrightarrow{d_1} \widetilde{H}_0(U \cap V) \xrightarrow{g_0} \widetilde{H}_0(U) \oplus \widetilde{H}_0(V) \xrightarrow{h_0} \widetilde{H}_0(X) \to 0$$

Notice that the last three terms are 0-th reduced homology groups. The same statement holds if we replace $\widetilde{H}_0$ with H_0. We can be more specific about the maps g_i and h_i. Let $\mathfrak{i} : U \to X$, $\mathfrak{j} : U \to X$, $\mathfrak{k} : U \cap V \to U$ and $\mathfrak{l} : U \cap V \to V$ be the inclusion maps and $\mathfrak{i}_* : H_i(U) \to H_i(X)$, $\mathfrak{j}_* : H_i(U) \to H_i(X)$, $\mathfrak{k}_* : H_i(U \cap V) \to H_i(U)$ and $\mathfrak{l}_* : H_i(U \cap V) \to H_i(V)$ are the induced map at the level of homology. Then we have:

$$g_i(\alpha) = (\mathfrak{k}_*(\alpha), \mathfrak{l}_*(\alpha)) \qquad h_i(\beta, \sigma) = \mathfrak{i}_*(\beta) - \mathfrak{j}_*(\sigma)$$

The following proposition asserts that the first homology group is determined by the fundamental group. In fact, it gives a rigorous meaning to the statement that the first homology group is the abelianization of the fundamental group:

Proposition 3.2. *Suppose X is a path connected space. The homology group $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$. In fact, if $\Phi : \pi_1(X) \to H_1(X)$ is the map which maps a closed loop γ based at the base point of X to the element of $H_1(X)$ which is given by γ, then Φ induces an isomorphism from $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ to $H_1(X)$.*

An n-dimensional *manifold* is a Hausdorff space which is locally homeomorphic to $\mathbf{R}^n$. The surface of g that you saw in the last class is an example of a 2-dimensional manifolds. More generally, we can consider manifolds with boundary which are Hausdorff topological spaces locally homeomorphic to open subspaces of the upper half-space $\mathbf{R}^{\geqslant 0} \times \mathbf{R}^{n-1}$. The homology groups of manifolds are more constrained than those of arbitrary topological spaces. For example, we have the following theorem:

Proposition 3.3. *If X is an n-dimensional manifold (possibly with boundary), then all homology groups $H_i(X)$ with $i > n$ are trivial. If X has non-empty boundary, then $H_n(X)$ is also trivial.*

Example 3.4. Let X be a space which has only one point. Since X is path connected, $H_0(X) = \mathbf{Z}$. Proposition 3.3 implies that all higher homology groups are trivial. Therefore, $H_i(X)$, for a contractible space X, is non-trivial only if $i = 0$, in which case it is equal to $\mathbf{Z}$.

Example 3.5. Propositions 3.2 and 3.3 imply that the homology groups of S^1 are given as follows:

$$H_i(S^1) = \begin{cases} \mathbf{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases} . \tag{3.6}$$

More generally, we can use the Mayer-Vietoris sequence to show that:

$$H_i(S^n) = \begin{cases} \mathbf{Z} & i = 0, n \\ 0 & i \neq 0, n \end{cases} . \tag{3.7}$$

Example 3.8. If K is a knot, then $H_1(S^3 \backslash K) = \mathbf{Z}$ (or equivalently $H_1(X(K)) = \mathbf{Z}$). This follows from the Mayer-Vietoris sequence applied to the decomposition of S^3 as the union of $S^3 \backslash K$ and a regular neighborhood of K. An examination of this argument show that we can even pick a generator for $H_1(S^3 \backslash K)$. Let $D^2 \times \{\text{pt}\}$ be a disc in the regular neighborhood $D^2 \times S^1$ of K. Then the boundary of $D^2 \times \{\text{pt}\}$ gives a loop in $S^3 \backslash K$ which is a generator of $H_1(S^3 \backslash K)$. This loop, often denoted by μ, is called a meridian of the knot K.

Example 3.9. If Σ_g° is the (oriented) Riemann surface of genus g with 1-boundary component, then:

$$H_i(\Sigma_g^\circ) = \begin{cases} \mathbf{Z} & i = 0 \\ \mathbf{Z}^{2g} & i = 1 \\ 0 & i > 2 \end{cases} . \tag{3.10}$$

This follows from the observation from the last class that Σ_g° has the same homotopy type as the wedge of $2g$ copies of S^1. We can also compute all homology groups of Σ_g by the tools that we have at this point. Our computation of $\pi_1(\Sigma_g)$ from the last class and Proposition 3.2 show that $H_1(\Sigma_g) = \mathbf{Z}^{2g}$. We can also use the Mayer-Vietoris exact sequence and (3.10) to show that $H_0(\Sigma_g) = H_2(\Sigma_g) = \mathbf{Z}$.