Problem Set 11

1. Let $\pi : E \to B$ be a covering map. We also assume that E is simply connected, namely $\pi_1(E)$ is the trivial group. Show that any point x in B has a neighborhood U such that the inclusion map $i : U \to B$ induces the trivial map $i_* : \pi_1(U, x) \to \pi_1(B, x)$. That is to say any loop in U based at x is homologous to the trivial loop inside B. Any topological space with this property is called a *semi-locally simply connected* space.

2. Let B be a path connected and locally path connected topological space. Fix a base point b_0 for B and form the set:

$$\widetilde{X} := \{(x, [\gamma]) \mid x \in B,\ \ \gamma : [0,1] \to X \text{ is a continuous path from } b_0 \text{ to } x)\}$$

We say $(x, [\gamma])$ is equivalent to $(x', [\gamma'])$ if $x = x'$ and the closed path $\gamma * \overline{\gamma}'$, given by gluing γ to the reverse of γ', represents the trivial element in $\pi_1(B, b_0)$. Let X be the set of equivalence classes of this equivalence relation. If (x, γ) represents an element of X and U is an open neighborhood of x in B, then define:

$$V([x,\gamma], U) := \{[x', \gamma'] \in X \mid x' \in U,\ \gamma' = \gamma * \sigma \text{ where } \sigma \text{ is a path in } U \text{ from } x \text{ to } x'\}$$

(a) Show that the sets $V([x,\gamma], U)$ for all $[x,\gamma] \in X$ and U define a topology on X.

(b) Let $\pi : X \to B$ be the map that sends the class of $(x, [\gamma])$ to x. Show that π is continuous.

(c) Show that $\pi_1(X) = 1$.

(d) Suppose B is semi-locally simply connected. Show that π is a covering map.

3. (a) Suppose K_+ and K_- are two links related by a crossing change. Show that there are simple closed curves $\gamma_+ \subset S^3 \backslash K_+$ and $\gamma_- \subset S^3 \backslash K_-$ with $\mathrm{lk}(K_+, \gamma_+) = \mathrm{lk}(K_-, \gamma_-) = 0$ such that there is a homeomorphism $f : S^3 \backslash N(\gamma_+) \to S^3 \backslash N(\gamma_-)$ that maps K_+ to K_-. Here $N(\gamma_+)$ and $N(\gamma_-)$ are tubular neighborhoods of γ_+ and γ_-.

(b) Let U_1 be the unknot and K be an arbitrary knot. Show that for a positive integer n, there are disjoint simple closed curves $\gamma_1, \dots, \gamma_n \subset S^3 \backslash K$ and disjoint simple closed curves γ_1', $\dots$, $\gamma_n' \subset S^3 \backslash U$ such that each γ_i (resp. γ_i') has vanishing linking number with K (resp. the unknot U) and there is a homeomorphism

$$f : S^3 \backslash \bigcup_i N(\gamma_i) \to S^3 \backslash \bigcup_i N(\gamma_i')$$

that maps K to U. Here $N(\gamma_1), \dots, N(\gamma_n)$ are disjoint tubular neighborhoods of $\gamma_1, \dots, \gamma_n$, and $N(\gamma_1'), \dots, N(\gamma_n')$ are disjoint tubular neighborhoods of $\gamma_1', \dots, \gamma_n'$.

(c) Let $\Phi_0 : D^2 \times \mathbb{R} \to D^2 \times \mathbb{R}$ be the homeomorphism given by $\Phi_0(z, t) = (z, t + 1)$. Use the previous part to show that for any knot K, there is n and for $1 \leqslant i \leqslant n$ and $j \in \mathbb{Z}$, there is a closed loop $\gamma_{i,j}$ with disjoint open neighborhoods $N(\gamma_{i,j})$ such that:

(i) $\Phi_0(\gamma_{i,j}) = \gamma_{i,j+1}$,

(ii) The space $X_\infty(K)$ associated to K is obtained by removing the solid tori $N(\gamma_{i,j})$ and glue them back in a different way.