Problem Set 4

1. Verify various claims that we made about fundamental groups in the class. (You do not need to turn in your work on this problem. But if you haven't seen fundamental groups before, it is important to do this problem!)

2. Suppose p, q are coprime positive integers. Show that the complement of the torus $T_{p,q}$ has the same homotopy type as the quotient of $S^1 \times [0,1]$ with respect to the following action:

$$(\theta + \frac{1}{p}, 0) \sim (\theta, 0) \qquad (\theta + \frac{1}{q}, 1) \sim (\theta, 1).$$

   (Hint: See [Hat02, Example 1.24])

3. Let D be a connected diagram for a link L which is not the unknot. We regard D as a subspace of S^2. Suppose γ is a simple closed curve in S^2 which intersects D transversely at two points. By Jordan curve theorem the complement of γ in S^2 has two components U_1 and U_2, each of them homeomorphic to the 2-dimensional disc. The diagram D is prime if for any such choice of γ the pair $(U_i, U_i \cap D)$, for $i = 1$ or 2, is homeomorphic to the pair of a ball and a diagram for an unknotted arc in the ball. The diagram is *strongly prime* if for any such choice of γ, $U_i \cap D$ is the diagram of an arc in a ball with zero crossings. The goal of this problem is to show that if L has a connected, alternating and reduced diagram with n crossings, then any non-alternating prime diagram D of L has more than n crossings.

   (a) Let D be a reduced connected diagram. We color D in the chessboard fashion. Let Γ'_D be the graph that has black regions as its vertices and two vertices are connected to each other by an edge if the corresponding black regions share a crossing. Show that D does not have a cut edge, namely, removing interior of any edge of Γ'_D does not make the graph disconnected.

   (b) Show that Γ'_D does not have a cut vertex if and only if D is strongly prime. A cut vertex v is a vertex that we obtain a disconnected graph after removing v and the neighboring edges.

   (c) For this part and the next part, we assume that D is a connected, reduced, non-alternating and strongly prime. Given any edge of the graph Γ'_D, show that either removing the interior of the edge or shrinking the edge produces a graph without a cut vertex. Conclude that there is a crossing of D such that either its 0-resolution or 1-resolution produces a strongly prime non-alternating diagram.

   (d) Let m be the number of the crossings of D. By induction on m show that $|s_+(D)| + |s_-(D)| < m + 2$.

   (e) Conclude that if L has a connected, alternating and reduced diagram with n crossings, then any non-alternating prime diagram of L has more than n crossings.

References

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354 ↑1