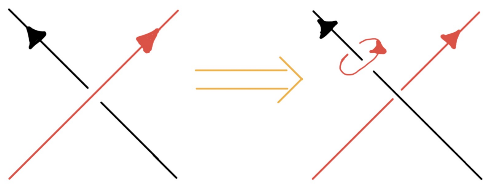
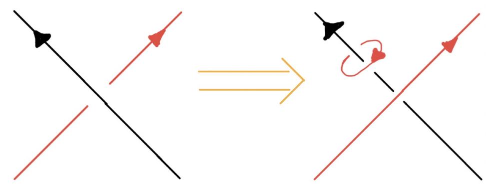
Problem Set 8

1. The goal of this problem is to show that the two definitions of linking number given in Problem Set 3 and Lecture 13 are the same. Suppose K_1 and K_2 are two oriented links in S^3, which are disjoint from each other. Then K_2 (resp. K_1) determines an element of the homology class of $H_1(S^3\backslash K_1)$ (resp. $H_1(S^3\backslash K_2)$). We also assume that a diagram $D_1 \cup D_2$ of $K_1 \cup K_2$ is given. We change K_2 in a neighborhood of a crossing as in Figures 1a or 1b to obtain K_2'.



(a)



(b)

Figure 1: In these figures, the black (resp. red) components are part of the diagram D_1 (resp. D_2) for the link K_1 (resp. K_2).

   (a) Show that K_2 and K_2' have the same homology class in $H_1(S^3\backslash K_1)$. In particular, the linking numbers $\mathrm{lk}(K_1, K_2)$ and $\mathrm{lk}(K_1, K_2')$, in the sense of Lecture 13, are equal to each other. (Hint: Recall that general properties of homology groups imply that you just need to find a map $f : \Sigma \to S^3\backslash K_1$ for an appropriate surface such that the boundary of Σ is equal to K_2 and $-K_2'$ where the latter is K_2' with the reverse orientation.)

   (b) Show that the linking numbers $\mathrm{lk}(K_1, K_2)$ and $\mathrm{lk}(K_1, K_2')$, in the sense of Problem Set 3, are equal to each other.

   (c) Show that if the diagrams D_1 and D_2 of K_1 and K_2 do not have any crossing with each other, then the linking numbers of K_1 and K_2, in both senses, are equal to zero.

   (d) Use the last three parts to show that the two definitions of linking number agree with each other. (Hint: See Problem 3 in Problem Set 2.)

2. Suppose M is a finitely generated abelian group. Determine all elementary ideals of M.

3. (a) Compute the Alexander polynomial of the figure-eight knot.

   (b) Compute the Alexander polynomial of the torus knot $T_{2,p}$.