Math Club Talk: 3/8/16 (1)

The Riemann zeta function and the distribution of primes.

Warning: I make no attempt to be rigorous. Do not believe any statement/equation I make! (But, the general procession of ideas should be correct.)

This talk is divided into 3 parts:

(1) Complex analytic properties of $\zeta(s)$ (No number theory.)

(2) Number theoretic properties of $\zeta(s)$ (P.N.T., R.H., explicit formula, etc.)

(3) An interpretation of R.H. via a probabilistic model

References:
- Complex Analysis, Stein, Shakarchi
- Multiplicative Number Theory, Davenport
- Multiplicative Number Theory, Montgomery, Vaughan

parts 1, 2

- Terry Tao's blog (part 3)

Part 1 : Define $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

since $\left|\frac{1}{n^s}\right| = \frac{1}{n^{Re(s)}}$, the sum converges on $\{Re(s) > 1\}$.

Fact: unif limit of holomorphic fn is holo

So $\zeta(s)$ is holo on $\{Re(s) > 1\}$.

Fact: "Analytic continuation"

A holomorphic fn can be extended in at most one way.

Extending $\zeta(s)$ to $\{Re(s) > 0\}$:

Main idea: compare sum to integral.

let $\delta_n(s) = n^{-s} + \int_n^{n+1} x^{-s}\,dx$

Then M.V.T. $\Longrightarrow$ $|\delta_n(s)| \leq \frac{|s|}{n^{Re(s)+1}}$ gained a power!

$$\sum_n \delta_n(s) = \zeta(s) + \underbrace{\int_1^{\infty} x^{-s}\,dx}_{= -\frac{1}{s-1}}$$

so $\zeta(s) = \underbrace{\sum_n \delta_n(s)}_{\text{holo. on } Re(s)>0} + \underbrace{\frac{1}{s-1}}_{\text{holo on } \mathbb{C}\setminus\{1\}}$

$\Rightarrow \zeta$ is holo on $\{Re(s)>0\}$ w/ a simple pole at $s=1$.

Next: extending $\zeta(s)$ to $\mathbb{C}$:

Define $\Gamma(s) = \int_0^\infty e^{-x} x^s \frac{dx}{x}$ Gamma fn $\Gamma(n) = (n-1)!$

"$\underbrace{\text{Mellin transform}}_{\text{Fourier transform on } (\mathbb{R}_+, \cdot)}$ of e^{-x}"

Define $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

"completed zeta fn"

Define $\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$ "Jacobi theta fn" "Fundamental sol'n to heat equation"

Then:
Poisson summation formula

$\Longrightarrow \vartheta(t) = t^{-1/2}\vartheta(t^{-1})$ "Modularity"

$\Longrightarrow \xi(s) = \xi(1-s)$ on $0 < Re(s) < 1$.

so we've extended ζ to $\mathbb{C}$!

Note: Γ has poles at $\{0, -1, -2, \dots\}$

$\Rightarrow$ ζ has zeros at $\{-2, -4, -6, \dots\}$
"trivial zeros"

This is as far as we'll get with complex analysis.

$\boxed{\text{Part 2:}}$

Fund. Thm. Arithmetic.
Expand the infinite product to get the equality

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{8^s} + \dots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \dots\right) \cdot \left(1 + \frac{1}{5^s} + \frac{1}{25^s} + \dots\right)$$

$$= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right)$$

$$= \prod_p (1 - p^{-s})^{-1}$$ "Euler product formula"

This is how ζ is connected to primes!

Consequences of Euler product:

1. $\exists$ ∞ly many primes.
   Pf. ζ has a pole at $s=1$.
   Let $s \downarrow 1$ in product formula.

   c.f. Dirichlet's theorem

(In fact, more careful analysis $\Rightarrow \sum \frac{1}{p} = \infty$)

2. ζ has no zeros in $\{Re(s)>1\}$

(and by symmetry no zeros in $\{Re(s)<0\}$

So all nontrivial zeros lie in the <u>critical strip</u> $\{0 \le Re(s) \le 1\}$.

For $s>1$,

$$\log \zeta(s) = \sum_p \log(1-p^{-s})^{-1}$$

$$= \sum_{p,m} \frac{p^{-ms}}{m}$$

$$\Longrightarrow \underbrace{(\log \zeta(s))'}_{\frac{\zeta'(s)}{\zeta(s)}} = \sum_{p,m} (-\log p)\, p^{-ms}$$

So $$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

$$\Lambda(n) = \begin{cases} \log p & \text{if } n=p^m \\ 0 & \text{otherwise.} \end{cases}$$

<u>Observe</u>: $\Lambda(n)$ is very similar to $\mathbb{1}_{prime}(n)$

except: "log p weights"

counts the prime powers as well

Define: $\pi(x) = \sum_{n \le x} \mathbb{1}_{\text{prime}}(n) = \left(\begin{array}{c}\#\text{ of primes}\\ \le x.\end{array}\right)$

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

Thm (Prime Number Theorem)

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1 \qquad \left(\begin{array}{c}\text{we write}\\ \pi(x) \sim \dfrac{x}{\log x}\end{array}\right)$$

Lem: PNT $\iff$ $\psi(x) \sim x$.

Pf: Let $\theta(x) = \sum_{p \le x} \log p$ $\quad \left(\begin{array}{c}\text{Not Jacobi}\\ \vartheta \text{ fn!}\end{array}\right)$

Then $\psi(x) = \theta(x) + O(x^{1/2})$ $\quad \left(\begin{array}{c}\text{throw away}\\ \text{terms}\end{array}\right)$

$$\pi(x) = \sum_{n \le x} \mathbb{1}_{\text{prime}}(n) = \sum \frac{\mathbb{1}_{\text{prime}}(n) \log n}{\log n}$$

integrate by parts (Riemann-Stieltjes) $\searrow$

$$= \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t}\, dt$$

If we replace $\theta(t)$ with t, we get $\int_2^x \frac{t}{t \log^2 t}\, dt$

(by parts again!) $\longrightarrow$ $= -\frac{x}{\log x} + \frac{2}{\log 2} - \int_2^x \frac{dt}{\log t}$

From this,

$$\pi(x) \sim \frac{x}{\log x} \iff \pi(x) \sim Li(x) := \int_2^x \frac{dt}{\log t}$$

$$\iff \theta(x) \sim x$$

$$\iff \psi(x) \sim x \qquad \square$$

Back to: $\psi(x) = \sum_{n \le x} \Lambda(n)$ $\qquad -\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$

<u>Thm</u>: $$\psi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s}{s}\left(-\frac{\zeta'}{\zeta}(s)\right) ds$$ (careful! not abs cvgt!)

<u>Pf</u>: contour integration, Mellin transform of $\mathbb{1}_{[0,x]}$, Mellin inversion formula. } just complex analysis $\square$

shift contour infinitely far to left. By residue formula,

$$\psi(x) = \underbrace{x}_{\text{from } s=1} - \underbrace{\underbrace{\frac{\zeta'(0)}{\zeta(0)}}_{\text{from } s=0}}_{\log 2\pi} - \underbrace{\underbrace{\sum_{k=1}^{\infty} \frac{x^{-2k}}{-2k}}_{\text{from trivial zeros}}}_{\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)} - \underbrace{\sum_{\rho} \frac{x^{\rho}}{\rho}}_{\text{from nontriv zeros}}$$

Not abs cvgt!

This is called the <u>explicit formula</u>.

So $\psi(x) = x - \underbrace{\sum \frac{x^\rho}{\rho}}_{\text{need to bound this.}} + O(1)$

Let $\Theta = \sup \{ \mathrm{Re}(\rho) \mid \rho \text{ zero of } \zeta \}$.
(Fact: $\frac{1}{2} \leq \Theta \leq 1$)
then $\psi(x) = x + O_\varepsilon(x^{\Theta + \varepsilon}) \quad \forall \varepsilon > 0$.

Riemann Hypothesis: $\Theta = \frac{1}{2}$.
Equivalently: $\psi(x) = x + O_\varepsilon(x^{1/2 + \varepsilon}) \quad \forall \varepsilon > 0$.
Equivalently $\pi(x) = Li + O_\varepsilon(x^{1/2+\varepsilon}) \quad \forall \varepsilon > 0$
("random walk", "square root cancellation")
We don't know anything nontrivial about Θ! (i.e. all we know is $\Theta \leq 1$).

We do know ζ has no zeros on $\mathrm{Re}(s) = 1$,
which gives
$$\psi(x) = x + O\left(x e^{-c\sqrt{\log x}}\right).$$
which is enough to imply P.N.T.

$\boxed{\text{Part 3:}}$ Why might we expect RH to be true?

Primes seem to "behave like random numbers."

Cramér random model for the primes
(See Terry Tao's blog; BTW, scroll down to the comments! You'll see evidence of me preparing last minute.)

Note: PNT $\Rightarrow$ "$\Pr(n \text{ is prime}) = \frac{1}{\log n}$"

Let's create a random set $\tilde{\wp}$ s.t.

$\forall n > 2$ $\Pr(n \in \tilde{\wp}) = \frac{1}{\log n}$, and indep over n,

ie. $\Pr(n_1, \ldots n_k \in \tilde{\wp}) = \frac{1}{(\log n_1) \cdots (\log n_k)}$

distinct

(Why does such a probability space exist? Kolmogorov extension theorem.)

We'll show Cramér random model "$\Rightarrow$ RH"

(Remark: Cramér random model $\Rightarrow$ twin primes
$\Rightarrow$ Goldbach
$\Rightarrow$ much more!)

Recall $\psi(x) = \sum_{(p,m):p^m \le x} \log p$.

Define $\tilde{\psi}(x) = \sum_{\substack{p,m \in \tilde{\mathcal{P}} \times \mathbb{N} \\ p^m \le x}} \log p$.

Then $\tilde{\psi}(x) = \underbrace{\sum_{\substack{p \in \tilde{\mathcal{P}} \\ p \le x}} \log p}_{\sum_{n \le x} \mathbb{1}_{\tilde{\mathcal{P}}}(n) \log n} + O(x^{1/2})$ $\left(\begin{array}{l}\text{recall} \\ \psi(x) \\ = \theta(x) + O(x^{1/2})\end{array}\right)$

Let $X_n = \mathbb{1}_{\tilde{\mathcal{P}}}(n) \log n - 1$. (random variable)

Then $\mathbb{E}[X_n] = 0$.

$\tilde{\psi}(x) - x = \sum_{n \le x} X_n \implies \mathbb{E}[\tilde{\psi}(x) - x] = 0$.

but what about the size of the error term. How large does it get?

(would like $\tilde{\psi}(x) = x + O_\varepsilon(x^{1/2+\varepsilon})$ $\forall \varepsilon > 0$)

Look at higher moments.

$$\mathbb{E}\Big[\Big(\sum_{n\le x} X_n\Big)^k\Big] = \sum_{n_1,\dots,n_k \le x} \mathbb{E}[X_{n_1} \cdots X_{n_k}]$$

Note: If $n_1 \neq n_j \ \forall j \geq 2$ then by independence: $\mathbb{E}[X_{n_1} \cdots X_{n_k}]$

$$= \underbrace{\mathbb{E}[X_{n_1}]}_{0} \, \mathbb{E}[X_{n_2} \cdots X_{n_k}] = 0.$$

So $$\mathbb{E}\Big[\Big(\sum_{n\le x} X_n\Big)^k\Big] = O_k\big(\underbrace{x^{k/2}}\ \underbrace{(\log x)^k}\big)$$

number of nonzero terms in sum

contribution of each term

Markov's inequality

$$\Rightarrow \forall k, \ \Pr\Big(\Big|\sum_{n\le x} X_n\Big| \geq t\Big) \leq \frac{1}{t^k} \mathbb{E}\Big[\Big|\sum X_n\Big|^k\Big]$$

Now given ε (as in page 10),

let $t = x^{\frac{1}{2}+\varepsilon}$ (since want $\tilde{\psi}(x) - x$ bounded by $O(x^{\frac{1}{2}+\varepsilon})$)

Then Markov

$$\Longrightarrow \forall k \quad \Pr\left(\left|\sum_{n\le x} X_n\right| \ge x^{\frac12+\varepsilon}\right)$$

$$\le \frac{1}{(x^{\frac12+\varepsilon})^k} O_k\left(x^{k/2}(\log x)^k\right)$$

$$= O_k\left(x^{-k\varepsilon}(\log x)^k\right).$$

(Fix)
Pick k so that $x^{-k\varepsilon}(\log x)^k = O_{\varepsilon}(x^{-2})$ (e.g.)

(This was the purpose of considering all k)

Thus, we've shown that:

For each x,
with probability $1 - O_\varepsilon(x^{-2})$

$$\sum_{n\le x} \mathbb{1}_{\tilde{p}}(n)\log n = x + O(x^{\frac12+\varepsilon})$$

(implied constant is 1!)

note the order!

Note $\sum_{x\in\mathbb{N}} x^{-2} < \infty$, so by Borel-Cantelli lemma,

With probability 1, for each x

$$\sum_{n\le x} \mathbb{1}_{\tilde{p}}(n)\log n = x + O_{\varepsilon,\tilde{p}}(x^{\frac12+\varepsilon})$$

so w.p. 1, $\tilde{\psi}(x) = x + O_{\varepsilon,\tilde{p}}(x^{\frac12+\varepsilon})$

i.e. random model satisfies RH!