Complex analysis

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Mathcamp 2020 Week 5

Contents

1 Introduction

1.1 Course blurb

We'll define what a contour integral in the complex plane is, and prove a nonempty subset of the following fundamental theorems from complex analysis: Cauchy's integral theorem, Cauchy's integral formula, analyticity of holomorphic functions, residue theorem.

1.2 Textbook reference

Stein and Shakarchi, *Complex analysis*

2 Day 1

2.1 Complex analysis is magic (a quick overview)

Many of the theorems we'll see deal with holomorphic functions and contour integrals.

First let's talk about holomorphic functions. A function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic* at z (or *complex differentiable* at z) if the following limit exists:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}. \tag{2.1}$$

Here the limit is taken over $h \in \mathbb{C}$.

Now, let's move on to contour integrals. First, a parametrized curve $\gamma : [a, b] \to \mathbb{C}$ is *continuously differentiable* or C^1 if $\gamma'(t)$ exists, is continuous, and $\gamma'(t) \neq 0$ for all t. In this course, we will assume all of our curves γ are piecewise C^1. If $f : \mathbb{C} \to \mathbb{C}$, we define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{2.2}$$

This is like the definition of the contour integral in multivariable calculus. To see the connection, you could also write

$$\int_\gamma f(z)\,dz = \int_\gamma f(x+iy)(dx+idy). \tag{2.3}$$

where the expression on the RHS is the usual line integral from multivariable calculus. Usually, in multivariable calculus, the contour integral contains an expression of the form $\vec{f}(\vec{x}) \cdot d\vec{r}$. But in complex analysis, we don't need a dot product. The expression $f(z)\,dz$ is just multiplication of complex numbers.

Here are some very nice theorems.

1. Contour integration (Cauchy's theorem): Suppose $\Omega \subset \mathbb{C}$ is an open set, and γ is a simple closed curve in Ω, and the region bounded by γ is also contained in Ω. If f is holomorphic on Ω,

$$\int_{\gamma} f(z)\, dz = 0 \tag{2.4}$$

2. Regularity and analyticity: If Ω is an open set, and f is holomorphic on Ω, then f is infinitely differentiable in Ω and f has a power series expansion everywhere.

3. Analytic continuation (principle of permanence): Let Ω be a connected open set. Suppose $w \in \Omega$ and (w_n) is a sequence in $\Omega \setminus \{w\}$ converging to w. Suppose f and g are holomorphic in Ω, and $f(w_n) = g(w_n)$ for all n. Then $f = g$ in all of Ω.

2.2 Holomorphicity

We already defined holomorphic above.

Example 2.1. The following functions are all holomorphic.

1. Polynomials are holomorphic on $\mathbb{C}$. Rational functions are holomorphic where they are defined.

2. Any function defined by a power series $\sum_{n=0}^{\infty} a_n z^n$ is holomorphic in its disc of convergence. For example, e^z and $\sin z$ are holomorphic on $\mathbb{C}$.

3. $\log z$ is well defined on the slit complex plane $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\}$. Then for any $\alpha \in C$, we can also define $z^\alpha = e^{\alpha \log z}$ on the slit complex plane.

Example 2.2. $f(z) = \overline{z}$ is not holomorphic. Neither is $f(z) = |z|^2 = z\,\overline{z}$. You are asked to verify these in the exercises. In general, when a complex conjugate appears, that is a sign that the function is not holomorphic.

Let's compare the concept of differentiability in various settings. Assume that $f(0) = 0$ and f is differentiable at 0:

1. Differentiability of $f : \mathbb{R} \to \mathbb{R}$ at the point 0:

$$f(h) = f'(0)h + \text{error}(h), \qquad \text{where } \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{|\text{error}(h)|}{|h|} = 0 \tag{2.5}$$

The term $f'(0)h$ is a product of two real numbers. This represents a rescaling of h

2. Holomorphicity (i.e., complex differentiability) of $f : \mathbb{C} \to \mathbb{C}$ at the point z:

$$f(h) = f'(0)h + \text{error}(h), \qquad \text{where} \lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{|\text{error}(h)|}{|h|} = 0 \tag{2.6}$$

   The term $f'(0)h$ is a product of two complex numbers. This represents a rotation and scaling of h.

3. Differentiability of $f : \mathbb{R}^2 \to \mathbb{R}^2$ at the point x:

$$f(h) = f'(0)h + \text{error}(h), \qquad \text{where} \lim_{\substack{h\to 0\\ h\in\mathbb{R}^n}} \frac{|\text{error}(h)|}{|h|} = 0 \tag{2.7}$$

   The term $f'(0)h$ is a product of a 2×2 matrix with a 2×1 matrix. This represents a linear transformation acting on h.

Given any $f : \mathbb{C} \to \mathbb{C}$, we can also convert it into a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ as follows. Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x + iy) = u(x, y) + iv(x, y)$. In other words,

$$u(x, y) = \operatorname{Re} f(x + iy) \tag{2.8}$$

$$v(x, y) = \operatorname{Im} f(x + iy) \tag{2.9}$$

Then we can set $F(x, y) = (u(x, y), v(x, y))$.

Example 2.3. Let $f(z) = z^3$. Note that $f(x+iy) = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$, so $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$.

If F is continuously differentiable, the derivative of F is the 2×2 matrix given by

$$F'(x, y) = \begin{pmatrix} \partial_x u(x, y) & \partial_y u(x, y) \\ \partial_x v(x, y) & \partial_y v(x, y) \end{pmatrix} \tag{2.10}$$

If $F(x, y)$ comes from a holomorphic function $f(z)$, then the following theorem tells us something about the matrix in (2.10).

Theorem 2.4. *Let $f : \mathbb{C} \to \mathbb{C}$. Let $u : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x + iy) = u(x, y) + iv(x, y)$. Suppose f is holomorphic at $z_0 = x_0 + iy_0$. Then*

$$f'(z_0) = \partial_x u(x_0, y_0) + i\partial_x v(x_0, y_0) \tag{2.11}$$

$$= \partial_y v(x_0, y_0) - i\partial_y u(x_0, y_0) \tag{2.12}$$

It follows that u and v satisfy the Cauchy-Riemann equations *at (x_0, y_0):*

$$\partial_x u(x_0, y_0) = \partial_y v(x_0, y_0) \tag{2.13}$$

$$\partial_x v(x_0, y_0) = -\partial_y u(x_0, y_0) \tag{2.14}$$

Proof. Suppose $f'(z_0)$ exists. Then

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{2.15}$$

If we let $h \to 0$ along the real axis (so $h = t$ where $t \in \mathbb{R}$ and $t \to 0$), then we get

$$\begin{aligned} f'(z_0) &= \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(z_0 + t) - f(z_0)}{t} && (2.16) \\ &= \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{[u(x_0 + t, y_0) - u(x_0, y_0)] + i[v(x_0 + t, y_0) - v(x_0, y_0)]}{t} && (2.17) \\ &= \partial_x u(x_0, y_0) + i \partial_x v(x_0, y_0) && (2.18) \end{aligned}$$

This proves (2.11). Next, if we let $h \to 0$ along the imaginary axis (so $h = it$ where $t \in \mathbb{R}$ and $t \to 0$), then we obtain (2.12). You are asked to check this in Exercise 3.4.

The Cauchy-Riemann equations follow by equating the real and imaginary parts of (2.11) and (2.12). □

One consequence of the Cauchy-Riemann equations (2.13) and (2.14) is the following: If $f'(z) = a + ib$ then

$$F'(x, y) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{2.19}$$

(Recall (2.10).) Note that matrix above is the 2×2 matrix representation of $a + ib$.

3 Day 1 exercises

Exercise 3.1. () In this problem, we will show that $f(z) = \overline{z}$ is not holomorphic in two different ways.

1. Use the definition of holomorphic. Start with the difference quotient

$$\frac{f(z + h) - f(z)}{h} = \frac{\overline{z + h} - \overline{z}}{h} \tag{3.1}$$

2. Use the Cauchy–Riemann equations.

Exercise 3.2. () Do the same as in Exercise 3.1, but with $f(z) = |z|^2$.

Exercise 3.3. () In each part below, you are given $f(z)$, which is a holomorphic function. Write down the functions $u(x, y)$ and $v(x, y)$ which satisfy $f(x + iy) = u(x, y) + iv(x, y)$, and check that the Cauchy-Riemann equations are satisfied.

1. $f(z) = z^3$
2. $f(z) = 1/z$
3. $f(z) = e^z$. (You will need the fact that $e^{i\theta} = \cos\theta + i\sin\theta$.)

Exercise 3.4. ( ) Finish the proof of Theorem 2.4. That is, show that $f'(z_0)$ is equal to (2.12).

Exercise 3.5. ( ) Let f be a holomorphic function, and let $f(x+iy) = u(x,y) + iv(x,y)$. Show that u satisfies the partial differential equation $(\partial_x)^2 u + (\partial_y)^2 u = 0$. A function satisfying this PDE is called a *harmonic function*. Show that v is harmonic as well.

Exercise 3.6. ( ) Let f be a holomorphic function, and let $f(x+iy) = u(x,y) + iv(x,y)$. Suppose we know $u(x,y) = -x^2 + y^2$. Can you determine v?

Exercise 3.7. ( ) Prove Cauchy's theorem. (You may assume that f' exists and is continuous.) Hint: Use the Cauchy–Riemann equations and Green's theorem.

4 Day 2

4.1 Cauchy's theorem

Cauchy's theorem, also known as the Cauchy–Goursat theorem, is the following.

Theorem 4.1 (Cauchy's theorem)**.** *Let Ω be an open set. Let f be holomorphic function on Ω. Let γ be a simple closed curve in Ω, such that the interior of Ω is contained in Ω. Then*

$$\int_\gamma f(z)\,dz = 0 \tag{4.1}$$

Proof. Let's prove Cauchy's theorem with the additional assumption that f' is continuous. (This assumption is not needed, but we would need a different proof.) We have

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)\,(dx+idy) = \int_\gamma (u+iv)\,dx + i(u+iv)\,dy \tag{4.2}$$

Since we assume f' is continuous, the partial derivatives of u and v exist and are continuous. By Green's theorem, the integral on the RHS of (4.2) is equal to

$$\iint_{\text{region bounded by } \gamma} i\frac{\partial}{\partial x}[u+iv] - \frac{\partial}{\partial y}[u+iv]\,dx\,dy, \tag{4.3}$$

which is equal to zero by the Cauchy-Riemann equations (2.13) and (2.14). □

Example 4.2. Here is a very important example. Let $f(z) = z^n$, where $n \in \mathbb{Z}$. Let $R > 0$ and let $C(0, R)$ denote the circle centered at the origin of radius R, oriented counter-clockwise. We can parametrize it with $\gamma : [0, 2\pi] \to \mathbb{C}$ be $\gamma(t) = Re^{it}$. From the definition of the contour integral, we can directly calculate that

$$\int_{C(0,R)} z^n \, dz = \int_0^{2\pi} \gamma(t)^n \gamma'(t) \, dt = \int_0^{2\pi} (Re^{it})^n \, (Rie^{it}) \, dt = \begin{cases} 0 & \text{if } n \neq -1 \\ 2\pi i & \text{if } n = -1 \end{cases} \tag{4.4}$$

For $n \geq 0$, we could have also used Cauchy's theorem to see that the integral is zero. For $n \leq -1$, Cauchy's theorem does not apply, because z^n is not holomorphic at 0.

Another way to see that the integral is 0 for $n \neq -1$ is to note that in that case, $z^n = F'(z)$, where $F(z) = \frac{z^{n+1}}{n+1}$. (We say that $F(z)$ is a *primitive* of z^n.) Then

$$\int_{C(0,R)} f(z) \, dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t) \, dt = \int_0^{2\pi} F'(\gamma(t))\gamma'(t) \, dt = \int_0^{2\pi} (F \circ \gamma)'(t) \, dt \tag{4.5}$$

$$= F(\gamma(2\pi)) - F(\gamma(0)) = F(R) - F(R) = 0 \tag{4.6}$$

However, that fact that the integral of $\int_{C(0,R)} z^{-1} \, dz \neq 0$ implies that z^{-1} does not have a primitive on $\mathbb{C} \setminus \{0\}$.

As shown in the example, $\gamma : [a, b] \to \mathbb{C}$ and F is a primitive of f, then

$$\int_\gamma f(z) \, dz = F(\gamma(b)) - F(\gamma(a)) \tag{4.7}$$

which is the complex version of the fundamental theorem of calculus for contour integrals.

4.2 Cauchy's integral formulas

Cauchy's theorem has the following consequence.

Theorem 4.3 (Cauchy's integral formula for f)**.** *Let Ω be an open set. Suppose f is holomorphic on Ω. Let D be an open disk such that $\overline{D} \subset \Omega$. ($\overline{D}$ denotes the closed disk.) Then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta \qquad \textit{for all } z \in D. \tag{4.8}$$

where C denotes the boundary of D, oriented counter-clockwise.

Proof. Note that $\frac{f(\zeta)}{\zeta - z}$ (as a function in ζ, with z fixed) is holomorphic in $\Omega \setminus \{z\}$. Apply Cauchy's theorem to this function with a keyhole contour (see picture from Jamboard), and let the width of the corridor go to zero. It follows that

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{C(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} \, d\zeta \tag{4.9}$$

where $C(z, \varepsilon)$ denotes the boundary of disk centered at z of radius ε, oriented counter-clockwise. We can use the usual parametrization of $C(z, \varepsilon)$ to get

$$\frac{1}{2\pi i} \int_{C(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + \varepsilon e^{it})}{\varepsilon e^{it}} (\varepsilon i e^{it}) \, dt = \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{it}) \, dt \tag{4.10}$$

Since f is continuous at z, as $\varepsilon \to 0$, the RHS of (4.10) converges to $f(z)$. (This is because of uniform convergence of the integrand.) The result follows. □

Theorem 4.4 (Cauchy's integral formula for $f^{(n)}$). *Suppose f is holomorphic in Ω. Then for all $n \geq 0$, $f^{(n)}$ exists on Ω, and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \qquad \text{for all } z \in D. \tag{4.11}$$

where D and C are as in Theorem 4.3

The idea of (4.11) is to differentiate (4.8) under the integral sign. We just need to do this rigorously. We will do that tomorrow.

5 Day 2 exercises

Exercise 5.1. () Let $f(z) = z^2 \cdot (\sin^3 z) \cdot e^{e^{z^4}} \cdot (z - 2020)^{-1}$. Let γ be the regular 2020-gon centered at the origin with a vertex at 1, oriented counter-clockwise. Evaluate $\int_\gamma f(z) \, dz$.

Exercise 5.2. () For $\xi > 0$ and $R > 0$, let $\gamma_{\xi,R}$ be the rectangle with vertices at $\pm R$ and $\pm R + i\xi$, oriented counter-clockwise. Evaluate $\int_{\gamma_{\xi,R}} e^{-\pi z^2} \, dz$.

Exercise 5.3. () Check that if you repeatedly differentiate (4.8) under the integral sign, you do in fact get (4.11).

Exercise 5.4. () Let γ be any closed curve in the plane that avoids the origin. (Here, γ is not required to be a simple curve. It is allowed to intersect itself.) What is a geometric description of the quantity $\frac{1}{2\pi i} \int_\gamma \frac{1}{z} \, dz$?

Exercise 5.5. () Show that for all $\xi \geq 0$,

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} \, dx = e^{-\pi \xi^2}. \tag{5.1}$$

In fact, (5.1) is also true for $\xi < 0$. This shows that the Fourier transform of $g(x) = e^{-\pi x^2}$ is $\widehat{g}(\xi) = e^{-\pi \xi^2}$.

Hint: Use Exercise 5.2. Also, use the fact that $\int_{-\infty}^{\infty} e^{-\pi x^2} \, dx = 1$. (This is known as the *Gaussian integral* and can be proved in many ways.)

Exercise 5.6. () Let $\Omega \subset \mathbb{C}$ be an open set containing 0. Suppose f is holomorphic in Ω except at the point 0. Furthermore, suppose that there is an open disc $D(0,r) \subset \Omega$ such that

$$f(z) = \sum_{n=-N}^{\infty} a_n z^n \qquad \text{for all } z \in D(0,r). \tag{5.2}$$

(Note the negative indices the sum. The sum in (5.2) is a *Laurent series*.)

Let D be an open disc such that $0 \in D$ and $\overline{D} \subset \Omega$. (Recall $\overline{D}$ denotes the closed disc.) Let C be the boundary of D, oriented counter-clockwise. Evaluate

$$\int_C f(z)\, dz. \tag{5.3}$$

Hint: You should use many of the ideas from today's class.

6 Day 3

6.1 Cauchy's integral formula (continued)

Now we give a proof of Theorem 4.4.

Proof. This is a proof by induction. We already know (4.11) for $n = 0$. Now let us prove it for $n = 1$ using the $n = 0$ case. The idea is to differentiate under the integral sign in a rigorous way. By (4.8),

$$f(z+h) - f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \left(\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z} \right) d\zeta \tag{6.1}$$

so

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)}\, d\zeta \tag{6.2}$$

Observe that

$$\lim_{h \to 0} \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)} = \frac{f(\zeta)}{(\zeta - z)^2} \qquad \text{uniformly for } \zeta \in C \tag{6.3}$$

so

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \tag{6.4}$$

which completes the proof for $n = 1$. To go from $n = k - 1$ to $n = k$ is similar. □

Corollary 6.1. *If f is holomorphic on Ω, then f is infinitely differentiable on Ω.*

Proof. This is part of the statement of Theorem 4.4. But I included it here to emphasize it. □

6.2 Fundamental theorem of algebra

The Cauchy integral formulas imply the fundamental theorem of algebra, by the following chain of theorems.

Corollary 6.2 (Cauchy inequalities)**.** *Suppose f is holomorphic in Ω. Let $D(z_0, R)$ be a disk such that $\overline{D(z_0, R)} \subset \Omega$. Then for all n,*

$$|f^n(z_0)| \leq \frac{n!}{R^n} \cdot \sup_{\zeta \in C(z_0,R)} |f(\zeta)| \tag{6.5}$$

Proof. Since $C(z_0, R)$ has length $2\pi R$, the Cauchy integral formula and the triangle inequality give us

$$|f^n(z_0)| = \left| \frac{n!}{2\pi i} \int_{C(z_0,R)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \right| \leq \frac{n!}{2\pi} \left(\sup_{\zeta \in C(z_0,R)} \frac{|f(\zeta)|}{|\zeta - z_0|^{n+1}} \right) 2\pi R. \tag{6.6}$$

Since $|\zeta - z_0| = R$, this completes the proof. □

Corollary 6.3 (Liouville's theorem)**.** *Suppose f is holomorphic on $\mathbb{C}$. (Such a function is called* entire*.) Suppose also that f is bounded, i.e., there exists $B > 0$ such that $\sup_{z \in \mathbb{C}} |f(z)| \leq B$. Then f is constant.*

Proof. Apply Corollary 6.2. Details left as an exercise. (The proof is very short.) □

Corollary 6.4 (Fundamental theorem of algebra)**.** *Any nonconstant polynomial has a root in $\mathbb{C}$.*

Proof. Apply Corollary 6.3. Details left as an exercise. (The proof is very short.) □

6.3 Analyticity

A function f is *analytic at z_0* if there exist a disc $D(z_0, R)$ and a sequence $(a_n)_{n=0}^\infty$ such that $f(z) = \sum_{n=0}^\infty a_n (z - z_0)^n$ for all $z \in D(z_0, R)$. A function f is *analytic in Ω* if it is analytic at every point in Ω.

Example 6.5. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{6.7}$$

The function f is infinitely differentiable. However, $f^{(n)}(0) = 0$ for all n, so f is not analytic at 0.

As Example 6.5 shows, for real functions, it is possible to be infinitely differentiable without being analytic. See https://en.wikipedia.org/wiki/Non-analytic_smooth_function for more.

So far, we have only shown that holomorphic functions are infinitely differentiable. But it turns out they are also analytic!

Theorem 6.6 (Analytic implies holomorphic). *Suppose $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$. Let R be the radius of convergence of the power series. Then f is holomorphic inside $D(z_0, R)$, and furthermore $f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$ inside $D(z_0, R)$.*

Proof. This follows from standard facts about power series from basic analysis. The proof uses no complex analysis, so we skip it. □

Theorem 6.7 (Holomorphic implies analytic). *If f is holomorphic in Ω, and D is an open disc centered at z_0 with $\overline{D} \subset \Omega$, then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \qquad \text{for all } z \in D \tag{6.8}$$

Proof. Let C denote the boundary of D. By the Cauchy integral formulas (4.11),

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \tag{6.9}$$

so

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_C f(\zeta)\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta \tag{6.10}$$

Note that by the geometric series formula,

$$\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} = \frac{1}{\zeta - z} \tag{6.11}$$

and furthermore, this convergence is uniform for $\zeta \in C$. Thus, we can interchange summation and integration to get that the RHS of (6.10) is equal to

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \tag{6.12}$$

By Cauchy's integral formula (4.8), this is equal to $f(z)$. □

Note that Theorem 6.7 also tells us about the radius of convergence of the Taylor series: If f is holomorphic on Ω, and $D(z_0, R) \subset \Omega$, then the radius of convergence of the Taylor series of f at z_0 is at least R.

Example 6.8. Consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{1+x^2}$. The Taylor series at 0 is $1 - x^2 + x^4 - x^6 + \cdots$. The radius of convergence of this is only 1, even though g is differentiable everywhere.

This is because the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = \frac{1}{1+z^2}$ is holomorphic except at $\pm i$. So by Theorem 6.7, the radius of convergence of the Taylor series of f at 0 must be 1.

7 Day 3

Exercise 7.1. () Check the details of the induction in Theorem 4.4. Show that the $n = k$ case follows from the $n = k - 1$ case.

Exercise 7.2. () Using the Cauchy inequalities (Corollary 6.2), prove Liouville's theorem (Corollary 6.3).

Exercise 7.3. () Using Liouville's theorem (Corollary 6.3), prove the fundamental theorem of algebra (Corollary 6.4).

Exercise 7.4. () Why isn't the function $\sin z$ a counterexample to Liouville's theorem?

8 Other topics

(We did not have time for these.)

8.1 Analytic continuation

We have the following theorem about uniqueness of analytic functions. This goes by the name of "analytic continuation" or "principle of permanence."

Theorem 8.1. *Let $\Omega \subset \mathbb{C}$ be an open connected set. Let $w \in \Omega$, and let $(w_k)_{k=1}^{\infty}$ be a sequence in $\Omega \setminus \{w\}$ that converges to w. If f is holomorphic in Ω, and $f(w_k) = 0$ for all k, then f is identically zero on Ω.*

Proof. Since f is analytic at w, there exists a disc $D \subset \Omega$ centered at w and a sequence $(a_n)_{n=0}^{\infty}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-w)^n \qquad \text{for all } z \in D. \tag{8.1}$$

We will first show that f is identically zero on D. Suppose for contradiction that this is not true. Then we can let m be the smallest index for which $a_m \neq 0$. For all $z \in D$,

$$f(z) = \sum_{n=m}^{\infty} a_n (z-w)^n = a_m (z-w)^m \left(1 + \frac{1}{a_m} \sum_{n=1}^{\infty} a_{m+n} (z-w)^n\right) \tag{8.2}$$

Let $g(z) = \frac{1}{a_m}\sum_{n=1}^{\infty} a_{m+n}(z-w)^n$ for $z \in D$. Observe that g is continuous on D and $g(w) = 0$ and.

$$f(z) = a_m(z-w)^m(1+g(z)) \tag{8.3}$$

If k is large enough, then $w_k \in D$ and $|g(w_k)| \leq \frac{1}{2}$. For such k, we have

$$0 = f(w_k) = a_m(w_k - w)^m(1+g(w_k)) \tag{8.4}$$

Since $w_k - w \neq 0$ and $1+g(w_k) \neq 0$, it follows that $a_m = 0$, which contradicts the assumption that f was identically zero on D.

From this, it is not hard to show that since Ω is connected, then f is identically zero on D. (This uses some topology. No analysis is needed.) □

Theorem 8.2. *Let $\Omega \subset \mathbb{C}$ be an open connected set. Let $w \in \Omega$, and let $(w_k)_{k=1}^{\infty}$ be a sequence in $\Omega \setminus \{w\}$ that converges to w. If f and g are holomorphic in Ω, and $f(w_k) = g(w_k)$ for all k, then $f = g$ on Ω.*

8.2 Residue theorem

Exercise 5.6 is closely related to what is called the "residue theorem." See https://en.wikipedia.org/wiki/Residue_theorem. The idea is that if you have $\int_\gamma f(z)\,dz$ where γ is a simple closed curve, but f has some singularities in the region bounded by γ, you can still in many cases evaluate this integral without too much work. You just need to calculate the "residue" at each singularity and sum them up.

The residue theorem gives you one way of evaluating integrals such as $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

8.3 Analytic number theory and the Riemann hypothesis

The field of analytic number theory uses analysis to study problems in number theory. For some reason, analytic number theorists use s for a complex number. Let's define the Riemann zeta function. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. This series converges to a holomorphic function on the right half-plane $\{s : \mathrm{Re}(s) > 1\}$. It turns out you can extend ζ to be a holomorphic function on $\mathbb{C} \setminus \{1\}$. (This is not at all trivial.) Because of analytic continuation, this extension is unique.

Define the prime counting function $\pi(x)$ to be the number of prime numbers $\leq x$. (This is standard notation.) There is an identity showing that a variant of $\pi(x)$ is equal to a contour integral where the integrand contains the expression $\frac{1}{\zeta(s)}$. Then by the residue theorem, this contour integral can be rewritten as a sum over the zeros of $\zeta(s)$. That's why the Riemann hypothesis is so important. For more information, see https://en.wikipedia.org/wiki/Explicit_formulae_for_L-functions#Riemann's_explicit_formula

8.4 Other topics

Mobius transformations, Riemann sphere, hyperbolic geometry