Dirac delta function

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Mathcamp 2020 Week 5

Contents

1 Introduction

1.1 Course blurb

The Dirac delta function, a.k.a. the unit impulse function, is the "function" which satisfies

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{1.1}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{1.2}$$

This may seem like nonsense, but this function shows up naturally in many physical problems.

In this class, we'll talk about the theory of distributions (note that "distribution" has many different meanings in mathematics), which will allow us to describe the delta function rigorously and make sense of statements such as $\frac{d^2}{dx^2}|x| = 2\delta(x)$. In fact, we'll learn how to differentiate *any* function. Then we'll see some applications of all this.

1.2 Reference

Terry Tao's blog post? But it might be too advanced. https://terrytao.wordpress.com/2009/04/19/245c-notes-3-distributions/

2 Day 1

2.1 The Dirac delta function

You may have heard of the Dirac delta function before. It is the "function" which satisfies

$$\delta_0(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{2.1}$$

and

$$\int_{-\infty}^{\infty} \delta_0(x)\, dx = 1 \tag{2.2}$$

and (a more general version of (2.2))

$$\int_{-\infty}^{\infty} f(x)\delta_0(x)\, dx = f(0) \qquad \text{for any continuous function } f. \tag{2.3}$$

Of course, $\delta_0(x)$ is not actually a function. It is what mathematicians call a "distribution" or a "generalized function." (Note that the word "distribution" has many meanings in mathematics.) An analysis professor from undergrad referred to δ_0 as the "mythical delta function."

Dirac introduced this in his 1930 textbook on quantum mechanics. (It showed up in other forms earlier as well.) Laurent Schwartz introduced the theory of distributions in 1945, which provided a framework for working with the Dirac delta function rigorously. This is kind of like the development of calculus. People were working with limits long before a rigorous definition was given.

One intuitive way to think about δ_0 is to consider, for $\varepsilon > 0$,

$$g_\varepsilon(x) = \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]} = \begin{cases} \frac{1}{2\varepsilon} & \text{if } |x| \le \varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

Then it is true, at least pointwise, that

$$\lim_{\varepsilon\to 0} g_\varepsilon(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases} \tag{2.5}$$

which can be a way to interpret (2.1). It is also true that for any continuous function f

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} f(x)g_\varepsilon(x)\,dx = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(x)\,dx = f(0)$$

which can be a way to interpret (2.3). See Exercise 3.8 for more on this.

2.2 Distributions

To understand distributions, we first a new way of thinking about functions. This approach can seem very abstract at first, but the benefit is that you can generalize the idea of derivative, and use these new ideas to help you solve differential equations. Today, we will focus on the theory. Tomorrow, we will focus on the applications.

We are used to thinking of a function $f : \mathbb{R} \to \mathbb{R}$ as something that takes in an input $x \in \mathbb{R}$ and outputs $f(x) \in \mathbb{R}$. The new way is to think of a function as operating on other functions. Given a function $f : \mathbb{R} \to R$, we would like to define a new operator

$$T_f : \{\text{functions } \mathbb{R} \to \mathbb{R}\} \to \mathbb{R} \tag{2.6}$$

by

$$T_f[\phi] = \int_{-\infty}^{\infty} \phi(x)f(x)\,dx. \tag{2.7}$$

In other words, we think of a function f in terms of what happens when you multiply by other functions and integrate. (I'll use square brackets when the operator takes a function as the argument. Also, "operator" is just a fancy word for function.)

Note that T_f is a *linear* operator, that is,

$$T_f[\phi + \psi] = T_f[\phi] + T_f[\psi] \text{ and } T_f[c\phi] = cT_f[\phi] \quad \forall c \in \mathbb{R} \text{ and } \forall \text{ functions } \phi \text{ and } \psi. \tag{2.8}$$

These follow from the properties of integration.

Actually, we need to be worry about covergence and integrability issues in (2.7), so we make the following definitions.

Definition 2.1. A function $\phi : \mathbb{R} \to \mathbb{R}$ is called a *test function* if it satisfies the following two properties.

1. It is infinitely differentiable.
2. It has *compact support.* (This means that there exists $R > 0$ such that for all $|x| > R$, $\phi(x) = 0$.)

The set of all test functions is denoted $C_c^\infty(\mathbb{R})$. (The "∞" refers to being infinitely differentiable. The "c" refers to having compact support.) Sometimes the set $C_c^\infty(\mathbb{R})$ is also denoted $\mathcal{D}(\mathbb{R})$.

Because $\phi \in C_c^\infty(\mathbb{R})$ is infinitely differentiable, we don't have to worry about issues of differentiability. Because it has compact support, we don't have to worry about issues of convergence of improper integrals.

Now we can define a distribution.

Definition 2.2. A *distribution* on $\mathbb{R}$ is a continuous linear operator $T : C_c^\infty(\mathbb{R}) \to \mathbb{R}$. The set of all distributions on $\mathbb{R}$ is denoted $\mathcal{D}'(\mathbb{R})$.

Recall the definition of linear in (2.8). I didn't define what it means for T to be continuous. Don't worry about it for now, but see See Exercise 3.9 if you would like to know more.

Remark 2.3. Clearly the constant function zero is in $C_c^\infty(\mathbb{R})$. If you would like an example of a nontrivial function, see Exercise 3.6. However, we will never need write down a test function explicitly. The reason is as follows. Whenever you think about a function like $f(x) = e^x$, you study properties of the function f as a whole. You usually don't need to evaluate f for specific values of f. Similarly, when we study a distribution $T[\phi]$, we don't need to evaluate T for specific test functions ϕ.

Fun fact 2.4. In complex analysis, the only infinitely differentiable function of compact support is the constant zero function. Fortunately, we are doing real analysis, and $C_c^\infty(\mathbb{R})$ contains many functions.

Example 2.5. Given any integrable function $f : \mathbb{R} \to \mathbb{R}$, we can define the distribution $T_f \in \mathcal{D}'(\mathbb{R})$ by

$$T_f[\phi] = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx \tag{2.9}$$

Note that this is well-defined for all $\phi \in C_c^\infty(\mathbb{R})$. We don't have to worry about convergence issues because ϕ has compact support. Also, T_f is linear, as noted previously.

Example 2.6. The Dirac delta function δ_0 is a distribution. It is defined by

$$\delta_0[\phi] = \phi(0) \tag{2.10}$$

This is clearly linear. Some people (especially physicists and engineers) like to write

$$\int_{-\infty}^{\infty} \phi(x)\delta_0(x)\, dx \tag{2.11}$$

when they mean $\delta_0[\phi]$ to make it look more like (2.9), even though δ_0 is not a function.

2.3 Derivatives

First, let's recall the integration by parts formula:

$$\int_a^b \phi(x)\psi'(x)\, dx = [\phi(x)\psi(x)]_a^b - \int_a^b \phi'(x)\psi(x)\, dx \tag{2.12}$$

Let $f : \mathbb{R} \to \mathbb{R}$. Suppose f is differentiable and f' is integrable. Can we find a connection between T_f and $T_{f'}$? We can use integration by parts:

$$T_{f'}[\phi] = \int_{-\infty}^{\infty} \phi(x) f'(x)\, dx \tag{2.13}$$

$$= [\phi(x) f(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \phi'(x) f(x)\, dx \tag{2.14}$$

$$= -\int_{-\infty}^{\infty} \phi'(x) f(x)\, dx \tag{2.15}$$

To get (2.15), recall that ϕ has compact support, so there boundary term $[\phi(x)f(x)]_{-\infty}^{\infty}$ is zero. Also, note that (2.15) is by definition equal to $-T_f[\phi']$. So we have

$$T_{f'}[\phi] = -T_f[\phi'] \tag{2.16}$$

Since we want to identify the function f and the operator T_f together, and we want to identify f' and $T_{f'}$, it makes sense to *define* $T_{f'}$ to be the derivative of the distribution T_f. This definition works in general.

Definition 2.7. Let T be a distribution. The *derivative* of T, denoted T', is the distribution given by

$$T'[\phi] = -T[\phi']. \tag{2.17}$$

This is also referred to as the *distributional derivative.*

Note that this definition works for *all* distributions. And since every function is a distribution, this definition works for *all* functions. In other words, this gives you a way to differentiate *all* functions.

Example 2.8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is integrable. The calculations shown above imply that $(T_f)' = T_{f'}$. (This is a nontrivial statement. Make sure you understand what this is saying.)

Example 2.9. Let $R(x)$ be the *ramp function.* It is defined by

$$R(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}. \tag{2.18}$$

We would like to show that the distributional derivative of R is the *Heaviside step function* $H(x)$, defined by

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}. \tag{2.19}$$

First,

$$T_R[\phi'] = \int_{-\infty}^{\infty} \phi'(x)R(x)\,dx = \int_0^{\infty} \phi'(x)x\,dx \tag{2.20}$$

By integration by parts,

$$\int_0^{\infty} \phi'(x)x\,dx = [\phi(x)x]_0^{\infty} - \int_0^{\infty} \phi(x)\,dx = 0 - \int_0^{\infty} \phi(x)\,dx = -T_H[\phi] \tag{2.21}$$

Thus, we have shown that

$$(T_R)'[\phi] = -T_R[\phi'] = T_H[\phi] \tag{2.22}$$

which shows that the distributional derivative of T_R is indeed T_H. We can write this as $(T_R)' = T_H$, or as $R' = H$.

Example 2.10. If $H(x)$ is the Heaviside step function (2.19), then $(T_H)' = \delta_0$. (Or, you can write $H' = \delta_0$.) You are asked to verify this in Exercise 3.1.

3 Day 1 exercises

Exercise 3.1. () Let $H(x)$ be the Heaviside function (see (2.19)). Show that $H' = \delta_0$. (Hint: Follow the same type of argument as in Example 2.9.)

Exercise 3.2. () What is δ_0'? What is $\delta_0^{(n)}$ (the nth derivative of δ_0)?

Exercise 3.3. () Let $f(x) = |x|$. Show that

$$f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \tag{3.1}$$

Then show that $f'' = 2\delta_0$.

Exercise 3.4. () For $a \in \mathbb{R}$, let δ_a be the distribution given by

$$\delta_a[\phi] = \phi(a). \tag{3.2}$$

Can you find a function f such that its distributional derivative is δ_a?

Exercise 3.5. () Let $f(x) = \lfloor x \rfloor$. What is f'? (That is, what is the distributional derivative of T_f?)

Exercise 3.6. () Can you give an example of a function in $C_c^\infty(\mathbb{R})$ that is not the zero function?

Hint: It might help to consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{3.3}$$

This function is infinitely differentiable on all of $\mathbb{R}$. (Can you show it?)

Exercise 3.7. () Let T_n be a sequence of distributions. We say that $T_n \to T$ (in the sense of distributions) if

$$\forall \phi \in C_c^\infty, \qquad T_n[\phi] \to T[\phi]. \tag{3.4}$$

Let $g_n(x) = \frac{n}{2}\mathbb{1}_{[-\frac{1}{n},\frac{1}{n}]}$. Show that $g_n \to \delta_0$ in the sense of distributions.

Exercise 3.8. () Let g be any function with $\int_{-\infty}^{\infty} |g(x)|\,dx < \infty$ and $\int_{-\infty}^{\infty} g(x)\,dx = 1$. Define $g_n(x) = ng(nx)$. Show that $g_n \to \delta_0$ in the sense of distributions.

Exercise 3.9. () Let ϕ_n be a sequence of test functions. We say that ϕ_n *converges to* ϕ *in* $C_c^\infty(\mathbb{R})$ if

1. There exists a $R > 0$ such that all the ϕ_n are supported in $[-R, R]$. (That is, for all n and for all $|x| > R$, $\phi_n(x) = 0$.)

2. For all n, $\phi^{(n)}$ converges uniformly to ϕ.

Now let $T : C_c^\infty(\mathbb{R}) \to \mathbb{R}$ be a linear operator. We say T is *continuous* if for all sequences ϕ_n converging to ϕ in $C_c^\infty(\mathbb{R})$, we have $T[\phi_n] \to T[\phi]$.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be integrable. Show that T_f is continuous.

2. Show that δ_0 is continuous.

3. Show that if T is continuous, then T' is continuous.

4 Day 2

Today we'll see some applications. We will be less rigorous than yesterday, although everything here can be made rigorous.

4.1 Convolution

The convolution of two functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\, dy \tag{4.1}$$

(Warning: The integral doesn't always converge. But if you restrict to "nice enough" functions, then it does. For example, the convolution of two test functions is another test function.)

The convolution can be thought of as a product of two functions. It satsfies some properties that the regular product satisfies, including $f * g = g * f$, $f * (g * h) = (f * g) * h$ and $f * (g + h) = f * g + f * h$. It also satisfies the following

Theorem 4.1. *For any two test functions ϕ and ψ, we have $(\phi * \psi)' = (\phi') * \psi = \phi * (\psi')$*

Proof. We have

$$\begin{aligned}
\frac{d}{dx}(\phi * \psi)(x) &= \frac{d}{dx}\int_{-\infty}^{\infty} \phi(y)\psi(x - y)\, dy & (4.2)\\
&= \int_{-\infty}^{\infty} \frac{d}{dx}[\phi(y)\psi(x - y)]\, dy & (4.3)\\
&= \int_{-\infty}^{\infty} \phi(y)\psi'(x - y)\, dy & (4.4)\\
&= (\phi * \psi')(x) & (4.5)
\end{aligned}$$

Differentiating under the integral sign is justified because ϕ and ψ are test functions. Since the convolution is commutative, we get the other desired equality as well. □

Example 4.2. As in (2.4), let $g_\varepsilon(x) = \frac{1}{2\varepsilon}\mathbb{1}_{[-\varepsilon,\varepsilon]}$. Then for any continuous function f, we have

$$(f * g_\varepsilon)(x) = \int_{-\infty}^{\infty} f(y) g_\varepsilon(x-y)\, dy = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(y)\, dy \tag{4.6}$$

so we have for each fixed x,

$$\lim_{\varepsilon\to 0}(f * g_\varepsilon)(x) = f(x) \tag{4.7}$$

Example 4.2 is closely related to the following result.

Theorem 4.3. *For any continuous function f, we have $f * \delta_0 = f$.*

Proof. Here is a non-rigorous proof: $(f * \delta_0)(x) = (\delta_0 * x) = \int_{-\infty}^{\infty} \delta_0(y) f(x-y)\, dy = f(x)$.

To prove this rigorously, we need to first define convolutions with distributions. It is not possible to define a convolution of two distributions in general, but it is possible to define a convolution of a distribution with a test function. See Exercise 5.1. □

4.2 Solving constant coefficient linear differential equations

Consider a constant coefficient linear differential operator, i.e, something like

$$L = a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} \cdots + a_0. \tag{4.8}$$

Then

$$Lu(x) = a_n u^{(n)}(x) + a_{n-1}u^{(n-1)}(x) + \cdots + a_1 u'(x) + a_0 u(x) \tag{4.9}$$

In this section we'll talk about how to solve differential equations of the form $Lu = f$ for the function u. For the equation $Lu = 0$, you just need to calculate the roots of the characteristic polynomial of L. Let's consider the case when f is not the zero function.

Consider the following differential equation.

$$Lu = \delta_0 \tag{4.10}$$

To make sense of this, we need to think of both sides as distributions. Since u is a function, it is also a distribution, so all of its derivatives are distributions. Since Lu is a linear combination of u and its derivatives, Lu is a distribution as well. So what (4.10) really means is

$$(Lu)[\phi] = \delta_0[\phi] \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}). \tag{4.11}$$

For example, if $Lu = u'' + 3u' + 2u$, then

$$(Lu)[\phi] = (T_u)''[\phi] + 3(T_u)'[\phi] + 2T_u[\phi] = \int_{-\infty}^{\infty} u(x)\left(\phi''(x) - 3\phi'(x) + 2\phi(x)\right)\,dx \tag{4.12}$$

(Note the negative sign in front of $\phi'(x)$. This is from the definition of distributional derivative.) Then (4.13) becomes

$$\int_{-\infty}^{\infty} u(x)\left(\phi''(x) - 3\phi'(x) + 2\phi(x)\right)\,dx = \phi(0) \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}). \tag{4.13}$$

A solution u to (4.10) is called a "fundamental solution" to the operator L. This is because if we want to solve $Lv = f$ for v, we can consider $v = u * f$. Then

$$Lv = L(u * f) = (Lu) * f = \delta_0 * f = f. \tag{4.14}$$

The fundamental solution u might be a distribution but not a function. This is not a problem. Even if u is a distribution, the convolution $u * f$ could be a function. But we have some issues with $u * f$.

1. The convolution $u * f$ might not be a distribution.
2. Or $u * f$ might be a distribution but not a function.
3. Or $u * f$ might be a function, but its kth derivative may not exist, even though L contains a kth derivative.

There are no quick solutions to these. There are many cases when you can get a distributional solution v to $Lv = f$. (This is also called a *weak solution.*) But then you have to do some work, or a lot of work, to show that v is a classical solution.

Example 4.4. Let's give a physical interpretation for a specific fundamental solution.

Consider $L = \frac{d}{dx}$, so $Lu = u'$. As we already saw, the Heaviside step function $H(x)$ (see (2.19)) is a fundamental solution to L. That is, $H' = \delta_0$. If we want to solve $v' = f$, we can let $v = H * f$ to get

$$v(x) = \int_{-\infty}^{\infty} f(y)H(x-y)\,dy = \int_{-\infty}^{x} f(y)\,dy \tag{4.15}$$

By the fundamental theorem of calculus, we do indeed have $v' = f$. However, note that the integral on the RHS of (4.15) might not converge, depending on what f is. (If f is a test function, then we are fine.)

Example 4.5. Let $Lu = u'' - u$. From classical mechanics,

$$x''(t) - x(t) = 0 \tag{4.16}$$

describes the *simple harmonic oscillator*, i.e. the oscillation of a mass on a spring with no other external forces. (This follows from Newton's second law and Hooke's law, and assuming both the mass and spring constant are 1.)

The general solution to (4.16) is $x(t) = A\sin t + B\cos t$, where A and B are arbitrary constants. The fact that the RHS is (4.16) means that there are no external forces acting on the mass-spring system.

If we had an external force, we would have $x''(t) - x(t) = f(t)$, where $f(t)$ is the external force at time t. The equation

$$x''(t) - x(t) = \delta_0(t) \tag{4.17}$$

would represent a very short punch at time $t = 0$. In this kind of context, δ_0 is called a unit impulse function. (Recall that impulse is the integral of the force over time, so δ_0 has a total impulse of 1.) One solution to (4.17) is

$$x(t) = \begin{cases} 0 & \text{if } t < 0 \\ \sin t & \text{if } t \geq 0 \end{cases} \tag{4.18}$$

(You are asked to verify this in Exercise 5.2.) To interpret (4.18), we have a mass at rest until a punch at time 0, which makes it start to oscillate. After the punch, it starts to oscillate. The function in (4.18) is called the *unit impulse response.*

4.3 Fourier transform and Laplace transform

In the previous section, I did not actually talk about how you find fundamental solutions. The Fourier transform and Laplace transforms are helpful in these types of problems.

The Fourier transform and Laplace transform of a function $f : \mathbb{R} \to \mathbb{R}$ are defined respectively by

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx \tag{4.19}$$

$$\mathcal{L}(f)(s) = \int_{0}^{\infty} f(x)e^{-sx}\,dx. \tag{4.20}$$

($\mathcal{F}(f)(\xi)$ is also denoted $\widehat{f}(\xi)$.) These are both useful tools in solving differential equations.

Then $\mathcal{F}(\delta_0)(\xi) = 1$ and $\mathcal{L}(\delta_0)(s) = 1$. The Fourier and Laplace transforms of f' also have very nice expressions:

$$\mathcal{F}(f')(\xi) = 2\pi i\xi\mathcal{F}(f)(\xi) \tag{4.21}$$

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0). \tag{4.22}$$

One way to solve a differential equation of the form $Lu = \delta_0$ is to take the Fourier transform or Laplace transform of both sides. See a differential equations textbook for more information.

4.4 Poisson's equation

Let $u : \mathbb{R}^n \to \mathbb{R}$. We can define the *Laplacian* of u, denoted Δu or $\nabla^2 u$, by

$$\Delta u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \tag{4.23}$$

Note that Δu is also a function $\mathbb{R}^n \to \mathbb{R}$. Δ is called the *Laplacian* or *Laplace operator*.

4.4.1 Motivation from physics

Let's see one example in physics where this operator shows up. Let $V : \mathbb{R}^3 \to \mathbb{R}$ be the electric potential. Then the electric field $E : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$E = -\nabla V. \tag{4.24}$$

By Gauss's law (one of Maxwell's equations), the charge density $\rho : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$\frac{\rho}{\varepsilon_0} = \nabla \cdot E, \tag{4.25}$$

where ε_0 is the vacuum permittivity constant.

If we combine the two equations, we get

$$-\Delta V = \frac{\rho}{\varepsilon_0} \tag{4.26}$$

4.4.2 Back to math

Poisson's equation is the following differential equation:

$$-\Delta u = f. \tag{4.27}$$

Theorem 4.6. *Let $n \in \mathbb{N}$. The fundamental solution to the Laplace operator in $\mathbb{R}^n$ is*

$$u(x) = \begin{cases} C_n |x|^{2-n} & \text{if } n = 1 \text{ or } n \geq 3 \\ -\frac{1}{2\pi} \log |x| & \text{if } n = 2 \end{cases} \tag{4.28}$$

where $C_n = \frac{1}{n(n-2)\alpha(n)}$ and $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$. That is, u satisfies

$$-\Delta u = \delta_0. \tag{4.29}$$

Proof. The proof is beyond the scope of this course. One way to prove this is to take the Fourier transform of both sides of (4.29). □

If $n = 1$, then $u(x) = -\frac{1}{2}|x|$, which agrees with Exercise 3.3.

If $n = 3$, then $u(x) = \frac{1}{4\pi|x|}$. Thus, if we have a point charge at the origin of charge $+1$, then the charge density is $\rho = \delta_0$, and the solution to (4.26) is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{1}{|x|}, \tag{4.30}$$

a well known formula for physics students. (The number $1/(4\pi\varepsilon_0)$ is known as Coulomb's constant.)

5 Day 2 exercises

Exercise 5.1. () Fact: It is possible to define the convolution of a distribution T and a test function ϕ. It turns out that $T * \phi$ is a function.

1. How should we define $T * \phi$?

   Hint: The definition of $T * \phi$ should be an extension of the definition of the convolution of two functions. That is, if f is an integrable function, then $T_f * \phi$ should equal $f * \phi$. (This is the same kind of reasoning we used to define the definition of the derivative of a convolution.)

2. Using this definition, check that $\delta_0 * \phi = \phi$ for any test function ϕ.

Exercise 5.2. () Verify that (4.18) satisfies (4.17). If you're not sure what you need to do, see the discussion around (4.13).

Exercise 5.3. () Open a differential equations textbook and learn about the Fourier transform and the Laplace transform!