Fourier analysis

Alan Chang

Mathcamp 2020 Week 3

Contents

1 Introduction

1.1 Course blurb

Around 1800, the French mathematician Jean-Baptiste Joseph Fourier accompanied Napoleon through Egypt. Egypt was very hot, and Fourier became interested in heat, so he developed Fourier series to solve the differential equation known as the "heat equation." (This is a story I heard from Elias Stein, the mathematician who taught me Fourier analysis.)

The central idea of Fourier series is to decompose a periodic function into pure oscillations (i.e., sine waves):

$$f(x) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{1.1}$$

This is what our ears do when we listen to music; it explains why the C-sharp of a piano sounds different from same C-sharp of a violin. (In class, we'll see this with some demonstrations using the software Audacity.)

Fourier analysis has wide applications to other areas, including signal processing (e.g., wireless communication), number theory (e.g., Dirichlet's theorem on primes in arithmetic progressions), quantum mechanics (e.g., the Heisenberg uncertainty principle, which Neeraja will cover in Week 4), and Boolean functions (as in Tim!'s Week 1 class).

In this class, we will learn how to find the Fourier series of any periodic function, prove some basic properties, and see how this can be used to solve differential equations. We will also look at the Fourier transform, which is an analogue of Fourier series for functions which are not periodic. With the remaining time, we'll discuss some of the many applications.

1.2 What Fourier analysis is not

https://www.smbc-comics.com/?id=2874

1.3 Warning about (lack of) rigor

Mathematical rigor will not be a focus of this course. Because we only have 5 days, if we tried to do everything rigorously, we would not have time to appreciate the beautiful theorems and applications of Fourier analysis. So instead, I'm going to give you some "proofs" that contain some unjustified steps, such as pretending that infinite sums work just like finite sums. Often, these incomplete/incorrect arguments are actually more useful in getting a feel for a subject. Recall that a large part of calculus was developed over hundred years before the modern definition of a limit was introduced.

If you still don't like this idea, I'd recommend taking a look at Terry Tao's blog post titled "There's more to mathematics than rigour and proofs": https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/

For those who want to practice writing rigorous analysis proofs, I will include some exercises on how to make certain arguments rigorous.

Fishy statements will be marked with a fish ().

1.4 Connections to other Mathcamp courses

Fourier analysis shows up in some form in the following classes (and perhaps more):

1. Week 1: Tim!'s class on boolean functions
2. Week 2: Neeraja's class on Weierstrass approximation
3. Week 2: Ben's class on the first uncountable ordinal
4. Week 4: Neeraja's class on the uncertainty principle

1.5 Reference

I highly recommend *Fourier analysis* by Stein and Shakarchi. Stein taught me Fourier analysis when I was at Princeton and his presentation was amazing. In fact, that was *the* class that made me start to consider specializing in analysis.

One great thing about the textbook is that it does not require much mathematical background – knowing some basic analysis would be enough. In contrast, most other books on the subject require at least measure theory and Lebesgue integration.

2 Day 1

2.1 The square wave

Definition 2.1. Let $L > 0$. A function $f : \mathbb{R} \to \mathbb{C}$ is *L-periodic* if $f(x + L) = f(x)$. (Note this means that every 1-periodic function is also 2-periodic.)

The statement "Let $f : \mathbb{R}/L\mathbb{Z} \to \mathbb{C}$" is essentially the same as saying "Let $f : \mathbb{R} \to \mathbb{C}$ be an L-periodic function." This is because you can think of $\mathbb{R}/L\mathbb{Z}$ as the set of real numbers, except you identify together two numbers if they differ by an integer multiple of L. (If this explanation doesn't make sense, don't worry about it.)

Let's start with a question: It is clear that a function like $5 + \cos x + \sin 2x - 4 \sin 3x$ is 2π-periodic. Given a 2π-periodic function $\mathbb{R} \to \mathbb{C}$, can we write it in the following form?

$$c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, \qquad \text{where } c_0, (c_n)_n, (b_n)_n \in \mathbb{C} \tag{2.1}$$

Let's jump straight into an example. Define $f : \mathbb{R} \to \mathbb{C}$ by first setting:

$$f(x) = \begin{cases} 1 & \text{if } x \in (0, \pi) \\ -1 & \text{if } x \in (-\pi, 0) \\ 0 & \text{if } x = -\pi, 0, \pi \end{cases} \tag{2.2}$$

and then extend f to a 2π-periodic function on $\mathbb{R}$. This function is called a square wave.

Let's *assume* that we can write f in the form (2.1). () If we make this assumption, can we figure out what the coefficients c_n, b_n must be?

First of all, since f is an odd function, all the c_n should equal 0, leaving us with

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{2.3}$$

Next, here is a crucial observation:

1. For all integers n, we have $\int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$.
2. For all integers m, n with $m \neq n$, we have $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$.

This means that if we multiply both sides of (2.3) by $\sin x$ and then integrate both sides, we should get ():

$$\int_{-\pi}^{\pi} f(x) \sin x \, dx = \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \sin x \, dx \tag{2.4}$$

$$= b_1 \int_{-\pi}^{\pi} \sin^2 x \, dx + \sum_{n=2}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin x \, dx \tag{2.5}$$

$$= \pi b_1 \tag{2.6}$$

(Warning: I did not justify why we could pull the infinite sum out of the integral. In general you cannot do this.)

So we get

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx. \tag{2.7}$$

By the same reasoning, we have, for all $n = 1, 2, \ldots$,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx. \tag{2.8}$$

Recalling the definition of f given in (2.2), these integrals are easy to evaluate. We get

$$b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{2.9}$$

so we end up with

$$f(x) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right) \tag{2.10}$$

If you plot some partial sums, you'll see that they do appear to get closer to the square wave. For example: https://www.wolframalpha.com/input/?i=4%2Fpi+%28+sin+x+%2B+1%2F3+sin+3x+%2B+1%2F5+sin+5x+%2B+1%2F7+sin+7x+%29

2.2 Demonstration of sound waves with software

For these demonstrations, start with low volume! There have been times when I accidentally generated a very loud tone while using earbuds.

2.2.1 Increasing the frequency until you hear a tone

Go here: https://www.szynalski.com/tone-generator/

1. Click on the sine wave on the bottom right corner and change it to "sawtooth."
2. Click on the frequency and chance it to 1 Hz.
3. (Make sure your volume is very low! And don't use headphones at first! You might hurt your ears if the sound is very loud!) Click on "Play." You should hear one click a second (since 1 Hz means one cycle per second).
4. Gradually increase the frequency. Eventually the clicks will sound like a very low tone.

2.2.2 Additive synthesis

To see how sine/cosine waves of different frequencies combine to form other waves: https://teropa.info/harmonics-explorer/ (You need to click "unmute" if you want to hear the sound.) One interesting thing you can do is start with the "base sine" wave (with the sound unmuted). Then click on "square" and "sawtooth." You can see (and hear!) the higher frequencies being added in!

2.2.3 Audacity

Another software is called Audacity. It is open source and cross platform. Here is an example of what you can do with Audacity. (I am using version 2.2.1.)

1. Generate → Tone. For waveform, choose Square. For frequency, put 220 Hz.
2. Select a part of the generated wave.
3. Analyze → Plot spectrum.
4. The Audacity documentation recommends using "Hann" for the function.

You should see the a chart with peaks at frequencies $\alpha, 3\alpha, 5\alpha, 7\alpha, \ldots$, where α is the fundamental frequency 220 Hz. (Note that the y-axis is measured in decibels, which is a logarithmic scale.)

You can also generate tones with other waveforms and see which frequencies have peaks. For example, if you choose a sine wave, there should be a single peak. You can also record your own sound files, e.g., tone of an instrument, you singing, random noise, and analyze those.

This page describes what the Plot Spectrum feature actually computes: `https://manual.audacityteam.org/man/plot_spectrum.html`

In music, when you have a complicated wave which is made up of sine waves of frequencies $\alpha, 2\alpha, 3\alpha, \ldots$, you can write this as $f(x) = \sum_{n=1}^{\infty} a_n \sin(2\pi n \alpha x)$. The lowest frequency α is called the fundamental frequency determines the pitch. The higher frequences $2\alpha, 3\alpha, \ldots$ are called higher harmonics or overtones. The amplitudes a_n determines the timbre of the sound. A middle-C square wave and a middle-C on a piano sound different because of how much of each higher harmonic is present in the sound wave.

Fun fact 2.2. Some basic waveforms are the square wave, pulse wave (a.k.a. rectangular wave), sawtooth wave, and triangle wave. See Exercise 3.1 for some of these waves. Sound processors of many early video game systems (e.g., NES) primarily generated these types of waves because they are very simple and require little memory. This gave well-known tunes such as the Super Mario Bros. theme their distinctive character.

2.3 Fourier series

(Note: We define $\sum_{n=-\infty}^{\infty} a_n$ to be $\lim_{N\to\infty} \sum_{n=-N}^{N} a_n$.)

Actually, (2.1) could be made cleaner if we introduce complex exponential functions. Recall Euler's formula: $e^{ix} = \cos x + i \sin x$. Using this, we have

$$c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=-\infty}^{\infty} a_n e^{inx} \tag{2.11}$$

where $(a_n)_{n=-\infty}^{\infty}, (b_n)_{n=1}^{\infty}, (c_n)_{n=0}^{\infty}$ satisfy

$$\begin{cases} c_0 = a_0 \\ c_n = a_n + a_{-n} & \text{for } n \geq 1 \\ b_n = i(a_n - a_{-n}) & \text{for } n \geq 1 \end{cases} . \tag{2.12}$$

Exercise 3.2 asks you to verify (2.12).

For $m, n \in \mathbb{Z}$, observe that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imx} e^{-inx}\, dx = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{2.13}$$

Protip 2.3. The constant i is just like any other constant. The usual derivative and integral rules apply. For example, $\frac{d}{dx} e^{inx} = ine^{inx}$ and $\int e^{inx}\, dx = \frac{e^{inx}}{in} + C$.

Now, suppose we have a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$. And let's *assume* that it can be written in the form $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$. Then, by the same (non-rigorous) reasoning as above with the sine function, we have $a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$

Definition 2.4. Let $f : [-\pi, \pi] \to \mathbb{C}$ be a Riemann integrable 2π-periodic function. The *Fourier coefficients* of f are the numbers $(a_n)_{n=-\infty}^{\infty}$ given by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx. \tag{2.14}$$

The *Fourier series* (or *trigonometric series*) of f is defined to be $\sum_{n=-\infty}^{\infty} a_n e^{inx}$. We write

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx} \tag{2.15}$$

to indicate that $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is the Fourier series of f.

Definition 2.5. A sum of the form $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ is called a *Fourier series* or a *trigonometric series*. A finite sum, e.g. $\sum_{n=-N}^{N} a_n e^{inx}$ is called a *trigonometric polynomial.*

Note that we don't write $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$. This is because we haven't actually proved that these two are the same function. In fact, there are situations where they aren't equal... See Section 2.5 for more information.

Fun fact 2.6. We can try to understand Fourier series from the point of view of linear algebra. Recall that the standard inner product (or dot product) of two vectors $\vec{u}, \vec{v} \in \mathbb{C}^n$ is given by

$$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} u_i \overline{v_i}. \tag{2.16}$$

If f and g are continuous functions $[-\pi, \pi] \to \mathbb{C}$, we can define an inner product via

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx. \tag{2.17}$$

As with vectors in $\mathbb{C}^n$, we say two functions are orthogonal if $\langle f, g \rangle = 0$. We can define the magnitude a function by $\langle f, f \rangle$.

Define $e_n(x) = e^{inx}$, so that (2.13) becomes

$$\langle e_m, e_n \rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}. \tag{2.18}$$

In linear algebra terms, the set $\{e_n : n \in \mathbb{Z}\}$ is an *orthonormal set*. That is, they all have magnitude 1 and are orthogonal to each other. Also, (2.14) can be rephrased as $a_n = \langle f, e_n \rangle$.

See Exercise 3.7 if you would like to review some properties of orthogonal vectors in a more familiar setting.

2.4 Uniqueness of Fourier series

Can two different functions have the same Fourier series? Here's a simple example to show the answer in general is "no." Let $f : [-\pi, \pi] \to \mathbb{C}$ be the constant 0 function and let $g : [-\pi, \pi] \to \mathbb{C}$ be defined by $g(0) = 1$ and $g(x) = 0$ otherwise. Then recalling (2.14), the Fourier coefficients of f and g are both $a_n = 0$.

But maybe that seemed like a silly counterexample. In fact, those are the only types of counterexamples, as the following theorem states.

Theorem 2.7. *Suppose f and g are functions with the same Fourier coefficients. If f and g are both continuous at x_0, then $f(x_0) = g(x_0)$.*

This immediately implies the following.

Corollary 2.8. *If f and g are continuous functions with the same Fourier coefficients, then $f = g$.*

Proof of Theorem 2.7. The proof is kind of technical, so we'll skip it. If you'd like to prove it yourself, see Exercise 3.9. □

2.5 Convergence of Fourier series

In general, convergence of Fourier series is a difficult topic. You can tell it is difficult based on how long the Wikipedia article is: https://en.wikipedia.org/wiki/Convergence_of_Fourier_series.

Suppose f is a continuous function and $f \sim \sum a_n e^{inx}$. Note that $|a_n e^{inx}| = |a_n|$. So if $\sum_{n=-\infty}^{\infty} |a_n| < \infty$, then by the Weierstrass M-test, the series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ converges uniformly. Note however that this argument does not imply that the series converges uniformly to f. But if f is continuous, then in fact it does. The extra ingredient you need is mean-square convergence ((2.19) below).

The previous paragraph also implies that if f has a jump discontinuity, then $\sum |a_n| = \infty$. (Exercise 3.11 asks you to work out the details.)

Here are some facts about convergence and non-convergence of Fourier series, all stated without proof. (In all of the following, f is 2π-periodic.)

1. Some good news:

   (a) If a function f is differentiable, then $\sum |a_n| < \infty$, and hence the Fourier series converges uniformly to f. See Exercise 3.10 for a slightly weaker statement. Also, see Fun fact 2.9 for a stronger statement.

   (b) If f is integrable, then we have mean-square convergence (a.k.a. L^2 convergence):

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(x) - \sum_{n=-N}^{N} a_n e^{inx} \right|^2 dx = 0 \tag{2.19}$$

   The proof of this is outlined in Exercise 5.21.

2. Some bad news:

   (a) There exists a continuous function f and a point x such that the Fourier series of f does not converge at x. Constructing a counterexample is very difficult. See Stein and Shakarchi, Fourier analysis, Chapter 3, Section 2.2.

The "good news" section above tells us in many cases, the Fourier series of f does indeed behave very nicely.

Fun fact 2.9. Actually, differentiability is sufficient but not necessary to conclude $\sum |a_n| < \infty$. The following is true.

1. A function is *α-Hölder continuous* if

$$\exists M \text{ s.t. } \forall x, \forall y, |f(x) - f(y)| \le M|x-y|^{\alpha}. \tag{2.20}$$

If f is α-Hölder continuous for some $\alpha > \frac{1}{2}$, then $\sum |a_n| < \infty$, and hence the Fourier series converges uniformly to f.

2. If f is α-Hölder continuous for some $0 < \alpha \leq \frac{1}{2}$, then the Fourier series converges uniformly to f (but it is not necessarily true that $\sum |a_n| < \infty$).

I actually was not aware of these facts until I started writing these notes. I found them at https://math.stackexchange.com/a/10816. This perhaps suggest that you don't need to worry about the details of convergence too much.

2.6 Basic properties of Fourier series

Here we prove some basic properties of Fourier series. I have included a "fishy proof" which is not quite rigorous, but gives you an idea of why something should be true.

Theorem 2.10. *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function with Fourier coefficients $(a_n)_n$. Suppose that f' exists and is integrable. Then f' has Fourier coefficients $(ina_n)_n$.*

Fishy proof. Suppose we have an equality ( ):

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}. \tag{2.21}$$

Now differentiate both sides. For the right side, differentiate term by term ( ).

$$f'(x) = \sum_{n=-\infty}^{\infty} \frac{d}{dx} a_n e^{inx} = \sum_{n=-\infty}^{\infty} ina_n e^{inx} \tag{2.22}$$

Thus, we see the Fourier coefficients of f' are $(ina_n)_n$. □

Real proof. Let b_n be the nth Fourier coefficient of f'. Then, by definition,

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx \tag{2.23}$$

Now integrate by parts to get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f'(x) e^{-inx}\, dx = \frac{in}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = ina_n \tag{2.24}$$

(I skipped some calculations.) □

The following theorem says that if you increase the speed of a sound wave by a factor of M, then the frequencies increase by a factor of M.

Theorem 2.11. *Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function with Fourier coefficients $(a_n)_n$. Let $M \in \mathbb{Z}_{>0}$ and set $g(x) = f(Mx)$. If $(b_n)_n$ denotes the Fourier coefficients of g, then*

$$b_n = \begin{cases} a_{n/M} & \text{if } M \mid n \text{ (i.e., if } M \text{ divides } n\text{)} \\ 0 & \text{otherwise} \end{cases} \tag{2.25}$$

Fishy proof. Suppose we have an equality (): $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$. Then

$$g(x) = f(Mx) = \sum_{n=-\infty}^{\infty} a_n e^{inMx} \tag{2.26}$$

which implies the Fourier coefficients of g are given by (2.25). □

Real proof. Exercise 3.8. □

3 Day 1 exercises

This class is "homework recommended." But there are several problems that you should do (or know how to do) so that you can get the most out of the lectures. You are definitely not expected to do all the problems – there are too many! I've included problems on many different topics and in many different difficulty levels, with the hope that every camper will find some interesting problems to think about.

3.1 Fourier series

Exercise 3.1. () This problem asks you to adapt the method we used in Section 2.1 to find the Fourier series for other waves.

1. Define the sawtooth wave by

$$f(x) = \begin{cases} x & \text{if } x \in (-\pi, \pi) \\ 0 & \text{if } x = -\pi, \pi \end{cases} \tag{3.1}$$

and extend to a 2π-periodic function on $\mathbb{R}$. By the same method we used for the square wave, find the Fourier series in sine-cosine form (2.1) for the sawtooth wave.

2. Define the triangle wave by

$$f(x) = |x| - \frac{\pi}{2} \qquad \text{if } x \in [-\pi, \pi] \tag{3.2}$$

Then extend it to be a 2π-periodic function. Find the Fourier series in sine-cosine form (2.1) for the triangle wave.

Also, some remarks.

1. You can use Wolfram Alpha to evaluate some of the integrals for you (e.g., `https://www.wolframalpha.com/input/?i=integrate+x*sin%28n*x%29+dx+from+-pi+to+pi`)

2. You can check your answer by plotting some partial sums.

3. Note that for both the square wave and the sawtooth wave, the sum of the coefficients does not converge absolutely. This is related to Exercise 3.11. However, for the triangle wave, the coefficients do converge absolutely. This is because the triangle wave is 1-Hölder continuous (a.k.a. Lipschitz continuous). See Fun fact 2.9.

Exercise 3.2. () Verify (2.12).

Exercise 3.3. () Let f be the square wave (see (2.2)). Calculate the Fourier coefficients a_n of f using the definition of Fourier coefficients (2.14). You should get:

$$a_n = \begin{cases} \frac{2}{in\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{3.3}$$

Check that this is consistent with (2.10) and (2.12).

Exercise 3.4. () Let f be the triangle wave (see (3.2)). Calculate the Fourier coefficients of f using any way you like. Here are some ways. (If you do at least two, make sure your answers agree!)

1. By the definition of Fourier coefficients (2.14).

2. By converting the sine/cosine series from Exercise 3.1 into the (complex) Fourier series with (2.12).

3. By Exercise 3.3 and Theorem 2.10.

Exercise 3.5. () Take (2.10) and plug in $x = \pi/2$. What do you get? (This is known as the *Leibniz formula for π*.) Can you discover anything else interesting by plugging in other values of x into any of the Fourier series we have calculated?

Exercise 3.6. () So far we've just been working with 2π-periodic functions. Suppose f is an L-periodic function instead. What should the analogue of Definition 2.4 be in this case? (Hint: most of it will look the same, except you will need to insert Ls in some places.)

Exercise 3.7. () Let $\vec{v}_1 = (1, 2, 1), \vec{v}_2 = (1, -1, 1), \vec{v}_3 = (1, 0, -1)$.

1. Verify that $\langle \vec{v}_1, \vec{v}_2 \rangle = \langle \vec{v}_1, \vec{v}_3 \rangle = \langle \vec{v}_2, \vec{v}_3 \rangle = 0$. (Recall that $\langle \cdot, \cdot \rangle$ denotes the inner product, a.k.a. dot product. For example, $\langle (1,2,3), (4,5,6) \rangle = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$.)

2. Suppose $(1,0,0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$ for some constants c_1, c_2, c_3. Show that $c_1 = 1/6$ without solving a system of linear equations. The technique here is similar to that of Section 2.1.

Exercise 3.8. ( ) Prove Theorem 2.11.

3.2 Uniqueness and convergence of Fourier series

Exercise 3.9. ( ) Suppose $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ is a function and all of its Fourier coefficients are 0, i.e.,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx = 0 \text{ for all } n \in \mathbb{Z}. \tag{3.4}$$

Show that if f is continuous at x_0, then $f(x_0) = 0$. (This implies Theorem 2.7.)

Hint: (This is an intentionally vague hint.) Suppose for contradiction that $f(x_0) \neq 0$. WLOG, assume $x_0 = 0$ and $f(0) > 0$. Choose $\varepsilon > 0$ appropriately. Then

$$\int_{-\pi}^{\pi} f(x)(\varepsilon + \cos x)^n\, dx = 0 \text{ for } n = 0, 1, 2, \ldots, \tag{3.5}$$

but also

$$\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x)(\varepsilon + \cos x)^n\, dx = \infty, \tag{3.6}$$

which is a contradiction.

Exercise 3.10. ( ) In this exercise we'll prove rigorously that if f'' is continuous, then the Fourier series of f converges. This is weaker than the stated result about differentiable functions stated in Section 2.5, but it's much easier to prove.

Suppose $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and f'' is continuous.

1. Prove that the Fourier coefficients of f satisfy

$$|a_n| \leq \frac{1}{n^2} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |f''(x)|\, dx \tag{3.7}$$

2. (This requires some analysis background.) Prove that the Fourier series of f converges uniformly.

Exercise 3.11. ( ) (This requires some analysis background.) Let f be a function with Fourier coefficients (a_n). Suppose f has a jump discontinuity. That is, suppose there is a point c such that $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$. Show that $\sum |a_n|$ diverges.

4 Day 2

4.1 Parseval's identity

Recall the following about vectors.

Theorem 4.1 (Pythagorean theorem). *If $(\vec{e}_n)_{n=1}^N$ are orthonormal unit vectors in $\mathbb{C}^N$, and $\vec{v} = \sum_{n=1}^N a_n \vec{e}_n$ for some $a_n \in \mathbb{C}$, then*

$$\|\vec{v}\|^2 = \sum_{n=1}^N |a_n|^2 \tag{4.1}$$

Proof. We can use the dot product of two vectors, which we'll denote $\langle \cdot, \cdot \rangle$.

$$\begin{aligned} \|\vec{v}\|^2 &= \langle \vec{v}, \vec{v} \rangle && (4.2)\\ &= \left\langle \sum_{n=1}^N a_n \vec{e}_n, \sum_{m=1}^N a_m \vec{e}_m \right\rangle && (4.3)\\ &= \sum_{n=1}^N \sum_{m=1}^N \langle a_n \vec{e}_n, a_m \vec{e}_m \rangle && (4.4)\\ &= \sum_{n=1}^N \sum_{m=1}^N a_n \overline{a_m} \langle \vec{e}_n, \vec{e}_m \rangle && (4.5) \end{aligned}$$

Since the vectors $(\vec{e}_n)$ are orthonormal, we have

$$\langle \vec{e}_n, \vec{e}_m \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \tag{4.6}$$

so in the double sum above, we only need to keep the terms where $n = m$. This gives

$$\|\vec{v}\|^2 = \sum_{n=1}^N |a_n|^2 \tag{4.7}$$

which completes the proof □

Now we can state Parseval's identity, which you can think of as the "Pythagorean theorem for Fourier series."

Theorem 4.2 (Parseval's identity). *If f is a continuous 2π-periodic function with Fourier coefficients $(a_n)_{n=-\infty}^{\infty}$,*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |a_n|^2 \tag{4.8}$$

Proof. () We can define an inner product on the space of continuous functions defined on $[-\pi, \pi]$. See Fun fact 2.6. Let $e_n(x) = e^{inx}$. Note that $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal set with respect to this inner product (see (2.18)).

The left hand side of (4.8) is $\langle f, f\rangle$. The Fourier series of f is $\sum_{n=-\infty}^{\infty} a_n e_n$. Now we can repeat the proof of Theorem 4.1.

$$\langle f, f\rangle = \left\langle \sum_{n=-\infty}^{\infty} a_n e_n, \sum_{m=-\infty}^{\infty} a_m e_m \right\rangle \tag{4.9}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n \overline{a_m} \langle e_n, e_m\rangle \tag{4.10}$$

$$= \sum_{n=-\infty}^{\infty} |a_n|^2 \tag{4.11}$$

That looks nice, but it was not a rigorous proof. There are some issues:

1. It is not necessarily true that a function f equals its Fourier series. So (4.9) is not justified.

2. You can't necessarily interchange sums and integrals when you have an infinite sum. So (4.10) is not justified.

If you want to fill in the gaps, you need to be more careful. See Exercise 5.21. □

The same type of orthogonality argument gives the following version o f Parseval's identity for two different functions.

Theorem 4.3 (Parseval's identity for two functions)**.** *If f and g are continuous 2π-periodic functions with Fourier coefficients $(a_n)_{n=-\infty}^{\infty}$ and $(b_n)_{n=-\infty}^{\infty}$ respectively,*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx = \sum_{n=-\infty}^{\infty} a_n \overline{b_n} \tag{4.12}$$

Note that Theorem 4.2 is just a special case of Theorem 4.3.

4.2 Convolutions

For two 2π-periodic functions f and g, we define their convolution by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy \tag{4.13}$$

In some sense, if we start with a function f and then convolve it with another function g, this can have an "averaging effect" on f. See Exercise 5.5 for more.

Here are some basic properties of convolutions. Taken together, they say that the convolution kind of behaves like a "product" of two functions.

Theorem 4.4 (Basic properties of convolutions). *Let f, g, h be any 2π-periodic functions and c any constant.*

1. $f * g = g * f$.
2. $f * (g + h) = f * g + f * h$.
3. $f * (cg) = c(f * g)$.

Proof. These follow from some basic properties of definite integrals. See Exercise 5.1. □

Now let $e_n(x) = e^{inx}$. The following suggests that convolutions will behave very nicely with Fourier series.

Theorem 4.5. *Let $m, n \in \mathbb{Z}$. Then*

$$e_m * e_n = \begin{cases} e_n, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \tag{4.14}$$

Proof. By the definition of convolution and some basic properties of integration,

$$(e_m * e_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e_m(y) e_n(x - y)\, dy \tag{4.15}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imy} e^{in(x-y)}\, dy \tag{4.16}$$

$$= e^{inx} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imy} e^{-iny}\, dy \tag{4.17}$$

Use (2.13) to finish the proof. □

The following very important theorem says that the convolution of two functions can be obtained by multiplying their Fourier coefficients together.

Theorem 4.6 (Convolution theorem for Fourier series). *Suppose $f \sim \sum a_n e^{inx}$ and $g \sim \sum b_n e^{inx}$. Then $f * g \sim \sum a_n b_n e^{inx}$.*

Fishy proof. (Note that this is very similar to the fishy proof of Theorem 4.2, except that we use (4.14) instead of (2.18).)

Let $f(x) = \sum_{n=-\infty}^{\infty} a_n e_n(x)$ and $g(x) = \sum_{m=-\infty}^{\infty} b_m e_m$ (). Then

$$f * g = \left(\sum_{n=-\infty}^{\infty} a_n e_n \right) * \left(\sum_{m=-\infty}^{\infty} b_m e_m \right) \tag{4.18}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_n b_m (e_n * e_m) \tag{4.19}$$

$$= \sum_{n=-\infty}^{\infty} a_n b_n e_n \tag{4.20}$$

where in the last line we used Theorem 4.5. Note that (4.19) involves interchanging an infinite sum and an integral ( ). But whatever, the proof is done. □

Real proof. Get ready for some double integrals. This is Exercise 5.4. □

If we have a function $f \sim \sum a_n e^{inx}$ and we want to remove all frequencies $|n| > N$, we can convolve f with $D_N(x)$, where

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}. \tag{4.21}$$

Observe that from the convolution theorem,

$$(f * D_N)(x) = \sum_{n=-N}^{N} a_n e^{inx} \tag{4.22}$$

In signal processing, this is an example of a low-pass filter: We start with a function f which has frequencies $n \in \mathbb{Z}$. Then by convolving with D_N, the resulting function $f * D_N$ only has "low" frequencies $|n| \leq N$. D_N is called the *Dirichlet kernel*. See Section 5.3 for some exercises about it.

4.3 Application: The Basel problem $(1 + \frac{1}{4} + \frac{1}{9} + \cdots)$

For some time, it was known that the series $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots$ converges. The *Basel problem*, posed in the 1600s, asks for the value of this series, and in the 1700s, Euler showed that the value is, surprisingly, $\frac{\pi^2}{6}$. Here we give a very short (but rigorous!) proof.

From Exercise 3.3, if f is the square wave (see (2.2)), then its Fourier coefficients are

$$a_n = \begin{cases} \frac{2}{in\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}. \tag{4.23}$$

Note that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 1 \tag{4.24}$$

and

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{8}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2}. \tag{4.25}$$

Thus, by Parseval's theorem, we have

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \tag{4.26}$$

From this, it follows (see Exercise 5.10) that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{4.27}$$

(Every step here was rigorous.)

5 Day 2 exercises

5.1 Convolutions

Recall the definition of the convolution of two functions given in (**??**).

Exercise 5.1. ( ) Prove Theorem 4.4.

Exercise 5.2. ( ) Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic function and with Fourier coefficients (a_n). Let $q(x) = e^{ikx}$. Show that $(f * q)(x) = a_k e^{ikx}$.

Exercise 5.3. ( ) Let $f : \mathbb{R} \to \mathbb{C}$ be any 2π-periodic function and let $q : \mathbb{R} \to \mathbb{C}$ be any trigonometric polynomial. (Recall the definition of trigonometric polynomial in Definition 2.5.) Show that $f * q$ is a trigonometric polynomial.

Exercise 5.4. ( ) Prove the convolution theorem (Theorem 4.6) rigorously.

Hint: The nth Fourier coefficient of $f * g$ is by definition

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (f * g)(x) e^{-inx}\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y)\, dy \right] e^{-inx}\, dx \tag{5.1}$$

Your goal is to show this is equal to $a_n b_n$, where a_n and b_n are the nth Fourier coefficients of f and g respectively. Try switching the order of integration and changing variables.

Exercise 5.5. ( ) This exercise is intended to help you gain some intuition about convolutions.

Let $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$. For $0 < r < \pi$, define a new function $f_r : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ by

$$f_r(x) = \frac{1}{2r} \int_{x-r}^{x+r} f(t)\, dt \tag{5.2}$$

In other words, the function $f_r(x)$ is defined as the average value of f on $[x-r, x+r]$. Here's the question: We can write $f_r = f * g_r$ for some function 2π-periodic $g_r : \mathbb{R} \to \mathbb{C}$. What is g_r?

Exercise 5.6. ( ) This is a continuation of Exercise 5.5. This problem requires some analysis background.

1. Suppose f is continuous. Show that $\lim_{r \to 0^+} f_r(x)$ converges pointwise to $f(x)$.
2. Even better: Suppose f is continuous. Show that $\lim_{r \to 0^+} f_r(x)$ converges uniformly to $f(x)$.

5.2 Consequences of Parseval's theorem

Exercise 5.7. ( ) Prove the Riemann-Lebesgue lemma: let $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ is any continuous function. Let $(a_n)_{n=-\infty}^{\infty}$ be the Fourier coefficients. Then $\lim_{n\to\infty} a_n = \lim_{n\to-\infty} a_n = 0$.

(The statement is true if we replace "continuous" with "Riemann integrable," but for that you'd need to use *Lebesgue's criterion for Riemann integrability*: https://en.wikipedia.org/wiki/Riemann_integral#Integrability. In fact, the statement is true for Lebesgue integrable functions as well, but for that you'd need to know about Lebesgue integration. In any case, our goal here is not to try to prove this in the greatest generality.)

Exercise 5.8. ( ) Suppose $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ is such that f' exists and is continuous. Let $(a_n)_{n=-\infty}^{\infty}$ be the Fourier coefficients of f. What is an identity relating $\int_{-\pi}^{\pi} |f'(x)|^2 \, dx$ and the numbers $(a_n)_{n=-\infty}^{\infty}$?

Exercise 5.9. ( ) Suppose $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ is such that f' exists and is continuous. Let $(a_n)_{n=-\infty}^{\infty}$ be the Fourier coefficients of f. Prove that $\sum |a_n|$ converges, and hence the Fourier series for f converges uniformly. This improves on Exercise 3.10.

Hint: The proof is very short (but not necessarily easy to find). You will need the following version of the Cauchy-Schwarz inequality:

$$\left(\sum_{n=-\infty}^{\infty} a_n b_n\right)^2 \leq \left(\sum_{n=-\infty}^{\infty} |a_n|^2\right)\left(\sum_{n=-\infty}^{\infty} |b_n|^2\right) \tag{5.3}$$

Exercise 5.10. ( ) Given that $\sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$, show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. Hint: Start with $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and rearrange the terms in the sum. (Rearranging terms is justified here! This is because all the terms are nonnegative.)

Exercise 5.11. ( ) Show that $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$.

Exercise 5.12. ( ) Let m be a positive integer. Can you find a formula for $\sum_{k=1}^{\infty} \frac{1}{k^{2m}}$?

5.3 The Dirichlet kernel

Recall the definition of Dirichlet kernel $D_N(x)$ given in (4.21).

Exercise 5.13. ( ) Show that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) \, dx = 1 \tag{5.4}$$

Exercise 5.14. ( ) Show that

$$D_N(x) = \frac{\sin((N + \frac{1}{2})x)}{\sin(\frac{x}{2})} \tag{5.5}$$

Hint: Geometric series

Exercise 5.15. () Using (5.5), show that for any $\delta > 0$,

$$\lim_{N\to\infty} F_N(x) = 0 \text{ uniformly on } \{x : \delta \le |x| \le \pi\}. \tag{5.6}$$

5.4 The Fejér kernel

The next few problems are about the Fejér kernel. For $N \in \mathbb{Z}_{\ge 0}$, define the Fejér kernel $F_N : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ by

$$F_N(x) = \sum_{|n|\le N-1} \left(1 - \frac{n}{N}\right) e^{inx} \tag{5.7}$$

You don't need to know about the Fejér kernel for this class, but it has some interesting properties. It functions as a low-pass filter, like the Dirichlet kernel $D_N(x)$ in (4.21). In some sense, the Fejér kernel is nicer because its Fourier coefficients decrease gradually to zero, whereas the Dirichlet kernel has an abrupt jump. You can learn more here: `https://en.wikipedia.org/wiki/Fej%C3%A9r_kernel`

Exercise 5.16. () Show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1 \tag{5.8}$$

Exercise 5.17. () Show that

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)} \tag{5.9}$$

Exercise 5.18. () Using (5.9), show that for any $\delta > 0$,

$$\lim_{N\to\infty} F_N(x) = 0 \text{ uniformly on } \{x : \delta \le |x| \le \pi\}. \tag{5.10}$$

Exercise 5.19. () Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous 2π-periodic function. Use (5.8), (5.9), and (5.10) to show that

$$f * F_N \to f \text{ uniformly as } N \to \infty. \tag{5.11}$$

Using (5.11) and Exercise 5.3, conclude that "trigonometric polynomials are dense in the space of continuous functions on $\mathbb{R}/2\pi\mathbb{Z}$." That is,

$$\forall \text{ continuous } f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}, \forall \varepsilon > 0, \exists \text{ a trig poly } q(x) \text{ s.t. } \sup_{x\in\mathbb{R}} |f(x) - q(x)| \le \varepsilon. \tag{5.12}$$

Friendly reminders:

1. Recall the definition of "trigonometric polynomial" in Definition 2.5.

2. "Sup" stands for "supremum." Look it up if you have not seen the definition before. The statement "$\sup_{x\in\mathbb{R}} |f(x) - q(x)| \le \varepsilon$" is equivalent to "$\forall x \in \mathbb{R}, |f(x) - q(x)| \le \varepsilon$."

The proof may be difficult if you have not seen this kind of argument before. The technique is related to what you should use in Exercise 5.6. Talk to me for some hints! (Neeraja covered this in her Week 2 class.)

Exercise 5.20. ( ) Fact: (5.11) is not true if we had the Dirichlet kernel $D_N(x)$ in place of the Fejér kernel $F_N(x)$. This is one reason convergence of Fourier series is a difficult topic.

What goes wrong with the proof you gave in Exercise 5.19?

5.5 Consequences of denseness of trigonometric polynomials

The following two exercises rely heavily on Exercise 5.19, but if you'd like you can take that exercise as given and see how it implies the results here.

Exercise 5.21. ( ) This exercise outlines a rigorous proof of both mean-square convergence (2.19) and Parseval's theorem (Theorem 4.2).

Recall the definition of the inner product $\langle f, g\rangle$ in (2.17), and define $\|g\| = \sqrt{\langle g, g\rangle}$. If f has Fourier coefficients $(a_n)_{n=-\infty}^{\infty}$, define $S_N(f)(x) = \sum_{n=-N}^{N} a_n e^{inx}$.

1. A trigonometric polynomial of the form $q(x) = \sum_{n=-N}^{N} b_n e^{inx}$ with $b_N \neq 0$ or $b_{-N} \neq 0$ is said to have degree N. Show that

$$\|f - S_N(f)\| \le \|f - q\| \text{ for all trigonometric polynomials } q \text{ of degree} \le N. \tag{5.13}$$

2. Using the previous part and (5.12), deduce that

$$\lim_{N\to\infty} \|f - S_N(f)\| = 0. \tag{5.14}$$

This completes the proof of mean-square convergence (2.19).

3. Show the following analogue of the Pythagorean theorem:

$$\|S_N\|^2 + \|f - S_N(f)\|^2 = \|f\|^2 \qquad \text{for all } N. \tag{5.15}$$

4. From the previous parts, deduce Parseval's theorem (Theorem 4.2).

Hint: It helps to think in terms of linear algebra. Think of $S_N(f)$ as the orthogonal projection of f onto the vector space spanned by the functions $\{e^{inx} : |n| \le N\}$.

Exercise 5.22. () Prove the Weierstrass approximation theorem:

$$\forall \text{ continuous } f : [-1,1] \to \mathbb{C}, \forall \varepsilon > 0, \exists \text{ a polynomial } p(x) \text{ s.t. } \sup_{x\in[-1,1]} |f(x) - p(x)| \le \varepsilon. \tag{5.16}$$

Here, $p(x)$ is a regular polynomial, not a trigonometric polynomial! Here are some hints:

1. Extend f to be a 2π-periodic function.
2. Show that trigonometric polynomials can be well approximated by polynomials on $[-1, 1]$.
3. Use (5.12).

Neeraja proved the Weierstrass approximation theorem in her Week 2 class by a different method. She used Bernstein polynomials.

6 Day 3

6.1 Application: Heat equation on the circle

First, let's introduce the *heat equation on the line*. Suppose we have an infinite metal rod, which we can think of as $\mathbb{R}$.

Let $u(x, t)$ be the temperature of the rod at point x at time t. The heat equation (which can be derived using physics) is a partial differential equation which says that

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t). \tag{6.1}$$

To see how solutions of the heat equation behave, check out `http://math.uchicago.edu/~luis/pde/heat.html`. (This app actually demonstrates the heat equation for a finite metal rod held at a constant temperature at both ends. See Exercise 7.2 for more.)

However, we've been working with periodic functions, so let's restrict ourselves to looking at functions $u(x, t)$ which are 2π-periodic in x. That is, $u(x + 2\pi, t) = u(x, t)$. To give a physical interpretation of periodic solutions, suppose we have a circular metal rod of radius 1, which we can think of as $\{(\cos x, \sin x) : x \in [-\pi, \pi)\}$. Suppose the initial temperature is $u(x, 0) = f(x)$ for some function $f : [-\pi, \pi) \to \mathbb{C}$, which we are given. We'd like to determine the temperature at future times. We can use Fourier series to do this.

Theorem 6.1 (Solution to heat equation)**.** *Let f is a 2π-periodic function with Fourier series $\sum_n a_n e^{inx}$. Suppose $u(x, t)$ satisfies the heat equation (6.1) and the initial condition*

$u(x,0) = f(x)$. *Then (under some mild smoothness conditions on f),*

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n e^{-n^2 t} e^{inx} \tag{6.2}$$

Proof. ( ) For each fixed t, we can think of $u(x,t)$ as a function in x, so we can expand it as a Fourier series in x:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{inx} \qquad \text{where } c_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x,t)e^{-inx}\,dx. \tag{6.3}$$

By differentiating with respect to x and t (illegally, since there is an infinite sum), we get

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=-\infty}^{\infty} c_n'(t) e^{inx} \tag{6.4}$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=-\infty}^{\infty} (-n^2) c_n(t) e^{inx} \tag{6.5}$$

Then by (6.1), and equating coefficients, we get

$$c_n'(t) = -n^2 c_n(t). \tag{6.6}$$

This is an ordinary differential equation! No more partial derivative nonsense. For each n, we can solve for $c_n(t)$ in this differential equation to get

$$c_n(t) = e^{-n^2 t} c_n(0). \tag{6.7}$$

By the initial condition $u(x,0) = f(x)$, we get $c_n(0) = a_n$. Thus,

$$c_n(t) = e^{-n^2 t} a_n \tag{6.8}$$

which completes the proof. □

Interpretation: If you split $u(x,t)$ up into different frequencies, the coefficient $e^{-n^2 t}$ tells you how quickly the nth frequency decays as the time t increases. In particular, the higher n is, the more quickly the frequency decays. This makes sense, temperature will "smooth out" over time and high frequencies are not smooth.

Fun fact 6.2. Define

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx}. \tag{6.9}$$

Then by the convolution theorem (Theorem 4.6), (6.2) could also be written

$$u(x,t) = (H_t * f)(x). \tag{6.10}$$

The function $H_t(x)$ is called the *heat kernel* (on the circle).

Define a rescaled version of the heat kernel as follows.

$$\Theta(z,\tau) = H_{-\pi i\tau}(2\pi z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}. \tag{6.11}$$

This function is called the *Jacobi theta function*. This function has many important applications in number theory, for example, in proving the functional equation for the Riemann zeta function (allowing you to extend $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ to all of the complex plane) and in proving the Jacobi four squares theorem (which gives you the exact number of ways any integer can be written as the sum of four squares).

Fun fact 6.3. The heat equation generalizes easily to d-dimensions. (For practical applications, perhaps $d = 3$ is the most important.) Let $\vec{x} \in \mathbb{R}^d$. Then the d-dimensional heat equation is:

$$\frac{\partial u}{\partial t}(\vec{x},t) = \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(\vec{x},t). \tag{6.12}$$

6.2 Application: The isoperimetric inequality

If a circle in $\mathbb{R}^2$ has area A and perimeter P, then $A = \frac{1}{4\pi}P^2$. The isoperimetric inequality states the following:

Theorem 6.4 (Isoperimetric inequality)**.** *For any shape in* $\mathbb{R}^2$*,*

$$\textit{area} \leq \frac{1}{4\pi}(\textit{perimeter})^2 \qquad \textit{with equality iff the shape is a circle.} \tag{6.13}$$

First we set up some background. A *parametrized curve* is a continuous function $\gamma : [a,b] \to \mathbb{R}^2$. A *curve* $\Gamma \subset \mathbb{R}^2$ is the image of a parametrized curve. A *simple closed* curve is a curve such that $\gamma(a) = \gamma(b)$ and γ restricted to $[a,b)$ is injective. That is, a simple closed curve is a curve which does not intersect itself and starts and ends at the same point.

Suppose $\gamma(t) = (x(t), y(t))$. We can think of $\gamma'(t) = (x'(t), y'(t))$ as the velocity vector of the curve. Then $\sqrt{|x'(t)|^2 + |y'(t)|^2}$ is the speed at time t. As a result,

$$\text{length of } \gamma = \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2}\, dt \tag{6.14}$$

Given any curve, we can parametrize it by arclength, which means we move along the curve at unit speed: $|x'(t)|^2 + |y'(t)|^2 = 1$ for all $t \in [a, b]$. In this case, the length of the curve is just $b - a$.

By Green's theorem (which is a special case of Stokes's theorem), if γ is a simple closed curve,

$$\text{area enclosed by } \gamma = \left| \int_a^b x(t) y'(t)\, dt \right| \tag{6.15}$$

We'll just take this as a fact. You are asked to verify this in Exercise 7.3.

Now that we have formulas for the length and area, we are ready to prove Theorem 6.4.

Proof of Theorem 6.4. Since we can rescale, we may assume without loss of generality that the length of γ is 2π. We may also assume that γ is the arclength parametrization. So $\gamma : [0, 2\pi] \to \mathbb{R}^2$, with $|x'(t)|^2 + |y'(t)|^2 = 1$ for all t. So (6.13) now becomes

$$\left| \int_0^{2\pi} x(t) y'(t)\, dt \right| \leq \pi \qquad \text{with equality iff } \gamma \text{ is a unit circle.} \tag{6.16}$$

Let's show (6.16) now. Let $x(t) \sim \sum a_n e^{int}$ and $y(t) \sim \sum b_n e^{int}$. Then $y'(t) \sim \sum inb_n e^{int}$, so by Parseval's identity (Theorem 4.3),

$$\frac{1}{2\pi} \int_0^{2\pi} x(t) y'(t)\, dt = \sum_{n=-\infty}^{\infty} a_n (\overline{inb_n}) \tag{6.17}$$

so by the triangle inequality,

$$\frac{1}{2\pi} \left| \int_0^{2\pi} x(t) y'(t)\, dt \right| \leq \sum_{n=-\infty}^{\infty} |n||a_n||b_n| \tag{6.18}$$

To motivate what happens next, our goal is to try to get to an expression involving $|x'(t)|^2 + |y'(t)|^2$. And recall that $x'(t) \sim \sum ina_n e^{int}$ and $y'(t) \sim \sum inb_n e^{int}$. So we should separate the $|a_n|$ and $|b_n|$ above. We can do this with the arithmetic mean–geometric mean inequality (or Cauchy-Schwarz): $|a_n||b_n| \leq \frac{|a_n|^2 + |b_n|^2}{2}$. So we have

$$\text{RHS of (6.18)} \leq \frac{1}{2} \sum_{n=-\infty}^{\infty} |n|(|a_n|^2 + |b_n|^2) \tag{6.19}$$

Next, we use $|n| \leq n^2$ so that we can apply Parseval again:

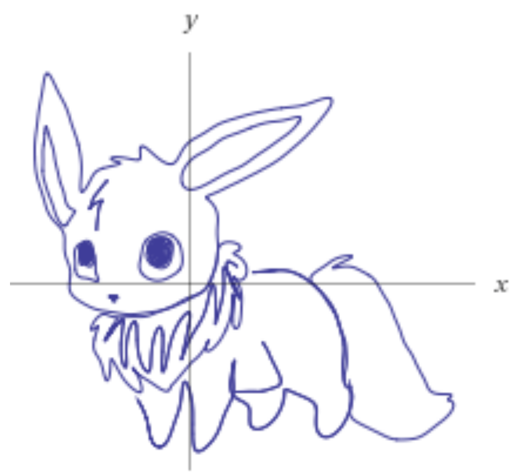
$$\text{RHS of (6.19)} \leq \frac{1}{2} \sum_{n=-\infty}^{\infty} |n|^2(|a_n|^2 + |b_n|^2) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (|na_n|^2 + |nb_n|^2) \tag{6.20}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (|x'(t)|^2 + |y'(t)|^2)\, dt \tag{6.21}$$

$$= \frac{1}{2} \tag{6.22}$$

where in the last line we used the fact that γ is parametrization by arclength. This chain of inequalities implies the inequality in (6.16). It remains to show that equality holds iff γ is a circle. That will be Exercise 7.4. $\square$

Fun fact 6.5. If you type "Eevee curve" on Wolfram Alpha (direct link: `https://www.wolframalpha.com/input/?i=eevee+curve`), you get the following



(plotted for t from 0 to 60π)

together with the parametric equations

- $x(t) = ((-\frac{8}{7}\sin(\frac{20}{13} - 27t) - \frac{9}{14}\sin(\frac{29}{19} - 25t) - \frac{2}{9}\sin(\frac{26}{17} - 16t) - \frac{13}{6}\sin(\frac{14}{9} - 15t) - \frac{39}{10}\sin(\frac{14}{9} - 11t) - \frac{41}{9}\sin(\frac{11}{7} - 10t) - \frac{19}{13}\sin(\frac{11}{7} - 6t) - \frac{9}{17}\sin(\frac{3}{2} - 5t) - \frac{135}{67}\sin(\frac{14}{9} - 4t) - \frac{5}{3}\sin(\frac{14}{9} - 2t) + \frac{1018}{11}\sin(t + \frac{11}{7}) + \frac{47}{7}\sin(3t + \frac{11}{7}) + \frac{9}{8}\sin(7t + \frac{8}{5}) + \frac{1}{8}\sin(8t + \frac{32}{7}) + 2\sin(9t + \frac{8}{5}) + \frac{2}{3}\sin(12t + \frac{51}{11}) + \frac{52}{17}\sin(13t + \frac{8}{5}) + \frac{2}{11}\sin(14t + \frac{9}{5}) + \frac{9}{11}\sin(17t + \frac{19}{12}) + 2\sin(18t + \frac{19}{12}) +$ `[50+ lines removed]`

- $y(t) = ((-\frac{11}{8}\sin(\frac{14}{9} - 27t) - \frac{30}{7}\sin(\frac{14}{9} - 25t) - \frac{5}{2}\sin(\frac{14}{9} - 23t) - \frac{54}{13}\sin(\frac{11}{7} - 21t) - \frac{147}{26}\sin(\frac{11}{7} - 19t) - \frac{41}{7}\sin(\frac{17}{11} - 17t) - \frac{30}{7}\sin(\frac{35}{23} - 14t) - \frac{94}{9}\sin(\frac{17}{11} - 11t) - \frac{65}{12}\sin(\frac{14}{9} - 10t) - \frac{14}{11}\sin(\frac{41}{27} - 9t) - \frac{5}{11}\sin(\frac{3}{2} - 5t) + \frac{644}{13}\sin(t + \frac{11}{7}) + \frac{543}{17}\sin(2t + \frac{11}{7}) + \frac{2}{11}\sin(3t + \frac{6}{5}) + \frac{35}{8}\sin(4t + \frac{11}{7}) + \frac{13}{9}\sin(6t + \frac{14}{9}) + \frac{3}{10}\sin(7t + \frac{55}{12}) + \frac{496}{55}\sin(8t + \frac{19}{12}) + \frac{51}{10}\sin(12t + \frac{11}{7}) +$ `[50+ lines removed]`

(I did not include the full equations because they would take up 2 full pages here.)

How is Wolfram Alpha generating these equations? It starts with a parametrization of the curve $\gamma : [0, 60\pi] \to \mathbb{R}^2$. Let $x(t)$ and $y(t)$ be the coordinate functions of $\gamma(t)$. Then Wolfram Alpha chops up $[0, 60\pi]$ into intervals of length 2π, and in each interval, it expands $x(t)$ and $y(t)$ as Fourier series. (That's what we did in our proof of the isoperimetric inequality!)

Instead of writing the Fourier series with complex exponentials, they use sines with shifted phases. These are equivalent because anything of the form $ae^{int} + be^{-int}$ can be rewritten in the form $c\sin(nt + d)$.

7 Day 3 exercises

7.1 Partial differential equations

Exercise 7.1. ( ) This problem is about the *wave equation on the circle.* The wave equation is

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{7.1}$$

Here is what wave equation (7.1) models: Imagine you have a Slinky (or a rope) which you can make oscillate up and down. Then $u(x,t)$ is the vertical displacement of the Slinky at position x at time t.

Now, let $u(x,t)$ be a 2π-periodic function in x, i.e., u satisfies $u(x+2\pi,t) = u(x,t)$. Suppose $u(x,t)$ satisfies the wave equation (7.1) with the initial conditions

$$u(x,0) = f(x) \sim \sum a_n e^{inx} \tag{7.2}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \sim \sum b_n e^{inx} \tag{7.3}$$

Express the solution $u(x,t)$ as a Fourier series. (Feel free to do illegal operations as we did in Theorem 6.1.) Can you find a physical/geometric description of the solution? Maybe you can say something about "traveling waves."

Because this problem deals with functions that are periodic in x, you can think of it as a giant Slinky wrapped around the equator. If we think of the equator as $\{(\cos x, \sin x) : x \in \mathbb{R}/2\pi\mathbb{Z}\}$, then $u(x,t)$ represents the displacement from equilibrium of the Slinky at $(\cos x, \sin x)$ at time t.

Exercise 7.2. ( ) This problem is about the heat equation. Consider a finite metal rod held at a constant temperature at both ends. We can think of the rod as the line segment $[0,\pi]$.

Let $f : [0,\pi] \to \mathbb{C}$ be a function with $f(0) = f(\pi) = 0$. Let $u(x,t)$ be a function, where $x \in [0,\pi]$ and $t \geq 0$. Suppose $u(x,t)$ satisfies the heat equation (6.1) with the initial condition

$$u(x,0) = f(x) \text{ for all } x \in [0,\pi] \tag{7.4}$$

and the boundary condition

$$u(0,t) = u(\pi,t) = 0 \text{ for all } t \geq 0. \tag{7.5}$$

(The boundary condition says that the rod is held at the constant temperature 0 at both ends.)

Solve for $u(x,t)$ in the following way: Extend f to be an odd function on $[-\pi,\pi]$, and then use our solution to Theorem 6.1. Why doesn't this work if we extend f to be an even function?

7.2 Isoperimetric inequality

Exercise 7.3. ( ) (This problem requires multivariable calculus background.) Prove the formula (6.15) we gave for the area enclosed by a curve γ.

Exercise 7.4. ( ) Complete the proof of the isoperimetric inequality (Theorem 6.4). That is, show that if area $\leq \frac{1}{4\pi}(\text{perimeter})^2$, then the shape is a circle. To do this, go back through the proof, and determine what happens if equality holds in every place that we wrote down an inequality sign.

7.3 Looking ahead: the Fourier transform

These exercises are to help you gain a little familiarity with the Fourier transform before we introduce it tomorrow.

We no longer consider periodic functions. For a function $f : \mathbb{R} \to \mathbb{C}$, define its Fourier transform by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\, dx. \tag{7.6}$$

(The symbole ξ is the Greek letter xi. Most mathematicians I know pronounce it like "ksee.") Also, define the convolution of two functions $f : \mathbb{R} \to \mathbb{C}$ and $g : \mathbb{R} \to \mathbb{C}$ by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\, dy \tag{7.7}$$

(This is the convolution that Roy encountered in his LaTeX typing game.) The integrals above are improper integrals (since they go from $-\infty$ to ∞), but for these exercises, you don't have to worry about that.

Exercise 7.5. ( ) Let

$$f(x) = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 1-|x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{7.8}$$

Calculate $\widehat{f}(\xi)$ and $\widehat{g}(\xi)$. You should get

$$\widehat{f}(\xi) = \frac{\sin \pi\xi}{\pi\xi} \qquad \text{and} \qquad \widehat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2. \tag{7.9}$$

(Remember that you can use Wolfram Alpha to calculate integrals for you!)

Exercise 7.6. ( ) Let f and g be as in Exercise 7.5. Compute the convolution $f * f$ and check that it is equal to g. (See (7.7) for the definition of convolution.)

Do you have a guess for what should be true about the Fourier transform of the convolution of two functions? That is, what should be the relationship between $\widehat{u}(\xi)$, $\widehat{v}(\xi)$, and $\widehat{u * v}(\xi)$?

Exercise 7.7. ( ) Let $f : \mathbb{R} \to \mathbb{C}$. Let $M > 0$ and set $g(x) = f(Mx)$. Show that $\widehat{g}(\xi) = \frac{1}{M}\widehat{f}(\frac{\xi}{M})$.

Hint: This should be very similar to the real proof of Theorem 2.11. By the definition of $\widehat{g}(\xi)$, we have

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i x\xi}\,dx = \int_{-\infty}^{\infty} f(Mx)e^{-2\pi i x\xi}\,dx. \tag{7.10}$$

Now do a change of variables.

Exercise 7.8. ( ) Show that

$$\widehat{\mathbb{1}_{[-r,r]}}(\xi) = \frac{\sin 2\pi r\xi}{\pi\xi}, \qquad \text{where} \qquad \mathbb{1}_{[-r,r]}(x) = \begin{cases} 1 & \text{if } |x| \leq r \\ 0 & \text{otherwise} \end{cases} \tag{7.11}$$

You can do this two different ways.

1. By the definition of the Fourier transform (8.4).

2. By Exercise 7.5 and Exercise 7.7.

8 Day 4

8.1 Fourier transform

Note: Neeraja's Week 4 class on the Heisenberg uncertainty principle will cover the Fourier transform in more detail and rigor than I do here.

Now we turn to functions $f : \mathbb{R} \to \mathbb{C}$ that are not periodic. We define the Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ of f by $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\,dx$.

Where does this definition come from? It turns it has some very nice properties similar to those of Fourier series. Here are some things we saw for Fourier series. Suppose f is a (nice) 1-periodic function. Then

1. Definition of Fourier coefficients:

$$a_n = \int_0^1 f(x)e^{-2\pi i n x}\,dx. \tag{8.1}$$

2. A reconstruction formula for f

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x} \tag{8.2}$$

3. Parseval's identity

$$\int_0^1 |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |a_n|^2 \tag{8.3}$$

When $f : \mathbb{R} \to \mathbb{C}$ is not periodic (but still nice), we have the following.

1. Definition of *Fourier transform* $\widehat{f} : \mathbb{R} \to \mathbb{C}$:

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} \, dx. \tag{8.4}$$

2. The *Fourier inversion formula*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi} \, d\xi. \tag{8.5}$$

3. *Plancherel's theorem*

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 \, d\xi \tag{8.6}$$

A few more things:

There's a version of Plancherel's theorem for two functions, which is the analogue of Theorem 4.3:

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\widehat{g}(\xi)} \, dx \tag{8.7}$$

Fun fact 8.1. There's an analogue of the orthogonality of $\{e^{inx} : n \in \mathbb{Z}\}$ (2.13), which is the following:

$$\int_{-\infty}^{\infty} e^{2\pi i \alpha x} e^{-2\pi i \beta x} \, dx \text{ ``$=$'' } \begin{cases} \infty & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases} \tag{8.8}$$

Maybe this seems like nonsense. When $\alpha \neq \beta$, the integral oscillates forever and doesn't converge, just like how $\int_{-\infty}^{\infty} \sin x \, dx$ doesn't converge.

But there are ways to interpret these integrals so that they do have a value of zero. To do this rigorously requires the mathematical theory of *distributions*. The *Dirac delta function* $\delta(x)$, also called the *unit impulse function* in signal processing, is an example of a distribution. The RHS of (8.8) is actually $\delta(\alpha - \beta)$. (When Elias Stein taught my Fourier analysis class, he once referred to $\delta(x)$ as the "mythical delta function.") See Exercise 9.5 for some calculations involving the delta function.

See https://en.wikipedia.org/wiki/Distribution_(mathematics).

8.2 Improper integrals and convergence issues

With the Fourier transform, we actually need to be even more careful than with Fourier series. Now we are integrating functions on the entire real line instead of the bounded interval $[-\pi, \pi]$, so we have to deal with improper integrals. In order for integrals like $\int_{-\infty}^{\infty} f(x)\,dx$ to converge, f needs to decay when x is large.

To avoid these kinds of convergence issues. We'll define $\mathcal{S}(\mathbb{R})$ to be, roughly speaking, the set of functions $f : \mathbb{R} \to \mathbb{C}$ which are infinitely differentiable and such that all of the derivatives of f decay very rapidly. Whenever I write "$f \in \mathcal{S}(\mathbb{R})$," it's enough to think "f is a very nice function and I don't have to worry about convergence issues." The actual definition of $\mathcal{S}(\mathbb{R})$ is given in (8.9).

Fun fact 8.2. Define the following two spaces of functions $f : \mathbb{R} \to \mathbb{C}$:

$$\mathcal{S}(\mathbb{R}) = \left\{ f \;\middle|\; \forall m, n \in \mathbb{Z}_{\geq 0}, \exists C_{m,n} \text{ s.t. } \forall x \in \mathbb{R}, |f^{(m)}(x)| \leq \frac{C_{m,n}}{|x|^n} \right\}. \tag{8.9}$$

An element of $\mathcal{S}$ is called a *Schwartz function*. The definition above says that a function $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function if it is infinitely differentiable and if all of its derivatives decay faster than every polynomial.

It may seem weird to focus on $\mathcal{S}(\mathbb{R})$. But the Fourier transform of a Schwartz function is again a Schwartz function, so $\mathcal{S}(\mathbb{R})$ is actually a very nice setting to study the Fourier transform. Furthermore, when we prove properties of the Fourier transform for Schwartz functions, we can often use "approximation arguments" to deduce these properties for certain other kinds of functions as well. This is a common technique in analysis.

8.3 Basic properties of the Fourier transform

The following is the analogue of Theorem 2.10. (We didn't cover this theorem in class.)

Theorem 8.3. *Let $f \in \mathcal{S}(\mathbb{R})$. Then the Fourier transfrom of f' is $\widehat{(f')}(\xi) = 2\pi i \xi \widehat{f}(\xi)$.*

Proof. By the Fourier inversion formula,

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{2\pi i x \xi}\, d\xi. \tag{8.10}$$

Now differentiate both sides. For the RHS, pass the derivative through the integral. (Since $f \in \mathcal{S}(\mathbb{R})$, this is perfectly justified.)

$$f'(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) \frac{d}{dx} e^{2\pi i x \xi}\, d\xi = \int_{-\infty}^{\infty} (2\pi i \xi) \widehat{f}(\xi) e^{2\pi i x \xi}\, d\xi. \tag{8.11}$$

Thus, we see that $\widehat{(f')}(\xi) = 2\pi i \xi \widehat{f}(\xi)$. □

The following is the analogue of Theorem 2.11.

Theorem 8.4. *Let $M > 0$ and set $g(x) = f(Mx)$. Then $\widehat{g}(\xi) = \frac{1}{M}\widehat{f}(\frac{\xi}{M})$.*

Proof. This was Exercise 7.7. □

8.4 Convolutions

Define the convolution of two functions by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy. \tag{8.12}$$

(Warning: We use the same symbol for convolution here as in the periodic function case, even though the definition is different in the two questions.)

There was a convolution theorem for Fourier series (Theorem 4.6). The following is the analogue for the Fourier transform.

Theorem 8.5 (Convolution theorem for the Fourier transform)**.** *Let $f, g \in \mathcal{S}(\mathbb{R})$. Then $\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.*

Proof. The real proof of this is very similar to the real proof of Theorem 4.6. We skip it. □

8.5 Application: The Borwein integrals

8.5.1 The setup

This purpose of this section is to see some examples of Fourier transforms and convolutions. The goal is to understand some very curious integrals. To clean up notation, let's define the following function

$$\operatorname{sinc} x = \frac{\sin x}{x}. \tag{8.13}$$

(The name of this function is pronounced like "sink." Also, $\operatorname{sinc} 0 = 1$.) Now observe the following, called the *Borwein integrals*.

$$\int_{-\infty}^{\infty} \operatorname{sinc} \pi t\,dt = 1 \tag{8.14}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}(\pi t)\,\operatorname{sinc}(\tfrac{\pi t}{3})\,dt = 1 \tag{8.15}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}(\pi t)\,\operatorname{sinc}(\tfrac{\pi t}{3})\,\operatorname{sinc}(\tfrac{\pi t}{5})\,dt = 1 \tag{8.16}$$

$$\vdots \tag{8.17}$$

$$\int_{-\infty}^{\infty} \operatorname{sinc}(\pi t)\,\operatorname{sinc}(\tfrac{\pi t}{3})\,\operatorname{sinc}(\tfrac{\pi t}{5})\,\cdots\,\operatorname{sinc}(\tfrac{\pi t}{13})\,dt = 1 \tag{8.18}$$

So far so good, but then

$$\int_{-\infty}^{\infty} \operatorname{sinc}(\pi t)\, \operatorname{sinc}(\tfrac{\pi t}{3}) \cdots \operatorname{sinc}(\tfrac{\pi t}{15})\, dt = \frac{935615849426881477393075728938}{935615849440640907310521750000} \tag{8.19}$$

$$\approx 1 - 1.5 \cdot 10^{-11} \tag{8.20}$$

Our goal in this section is to understand why this happens.

8.5.2 Fourier transform and the sinc function

First, let's understand a little more about the sinc function. As stated in Exercise 7.5 we have

$$\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}}(\xi) = \operatorname{sinc}(\pi\xi). \tag{8.21}$$

(See the exercise for the definition of $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$.) Now let's define some rescaled versions of $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}$. For $m > 0$, let

$$f_m(x) = m\mathbb{1}_{[-\frac{1}{2m},\frac{1}{2m}]}(x) = m\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(mx). \tag{8.22}$$

By Exercise 7.8, these functions satisfy

$$\widehat{f_m}(\xi) = \operatorname{sinc}(\tfrac{\pi\xi}{m}) \tag{8.23}$$

Now the Borwein integrals above can be rewritten

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, d\xi = 1 \tag{8.24}$$

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, d\xi = 1 \tag{8.25}$$

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, \widehat{f_5}(\xi)\, d\xi = 1 \tag{8.26}$$

$$\vdots \tag{8.27}$$

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, \widehat{f_5}(\xi) \cdots \widehat{f_{13}}(\xi)\, d\xi = 1 \tag{8.28}$$

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, \widehat{f_5}(\xi) \cdots \widehat{f_{13}}(\xi)\, \widehat{f_{15}}(\xi)\, d\xi < 1 \tag{8.29}$$

Let's deal with the first one (8.24). By the Fourier inversion formula,

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi) e^{2\pi i x\xi}\, d\xi = f_1(x). \tag{8.30}$$

Setting $x = 0$ gives which gives us (8.24).

Now let's move on to the second equation (8.25). For this one, we can use Plancherel's theorem for two functions (8.7) to get

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, dx = \int_{-\infty}^{\infty} f_1(x)\, f_3(x)\, dx \tag{8.31}$$

From the definitions of f_1 and f_3 (8.22), we have

$$\int_{-\infty}^{\infty} f_1(x)\, f_3(x)\, dx = \int_{-1/2}^{1/2} f_3(x)\, dx = 1, \tag{8.32}$$

which gives us (8.25).

Now we turn to the third equation (8.26). Unfortunately, there is no Plancherel for three functions. However, we can use the convolution theorem (Theorem 8.5), followed by Plancherel, followed by the definition of f_1, to get

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, \widehat{f_5}(\xi)\, d\xi = \int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3 * f_5}(\xi)\, d\xi \tag{8.33}$$

$$= \int_{-\infty}^{\infty} f_1(x)\, (f_3 * f_5)(x)\, dx \tag{8.34}$$

$$= \int_{-1/2}^{1/2} (f_3 * f_5)(x)\, dx \tag{8.35}$$

Note that this argument generalizes. For $k \geq 1$, then by repeatedly applying the convolution theorem,

$$\widehat{f_3 * f_5 * \cdots * f_{2k+1}}(\xi) = \widehat{f_3}(\xi)\, \widehat{f_5}(\xi)\, \cdots\, \widehat{f_{2k+1}}(\xi), \tag{8.36}$$

so the previous argument gives

$$\int_{-\infty}^{\infty} \widehat{f_1}(\xi)\, \widehat{f_3}(\xi)\, \widehat{f_5}(\xi)\, \cdots\, \widehat{f_{2k+1}}(\xi)\, d\xi = \int_{-1/2}^{1/2} (f_3 * f_5 * \cdots * f_{2k+1})(x)\, dx. \tag{8.37}$$

Now we just need to understand the RHS of (8.37) better.

8.5.3 More on convolutions

Theorem 8.6. *Let f and g be functions $\mathbb{R} \to \mathbb{C}$. Then*

$$\int_{-\infty}^{\infty} (f * g)(x) = \left(\int_{-\infty}^{\infty} f(x)\, dx\right) \left(\int_{-\infty}^{\infty} g(x)\, dx\right) \tag{8.38}$$

Proof. A natural way to do this is insert the definition of $f * g$ into the LHS of (8.38) and then interchange the order of integration.

But for fun, let's prove with the convolution theorem. By inserting $\xi = 0$ into the definition of the Fourier transform we have

$$\widehat{f}(0) = \int_{-\infty}^{\infty} f(x)\,dx \tag{8.39}$$

$$\widehat{g}(0) = \int_{-\infty}^{\infty} g(x)\,dx \tag{8.40}$$

$$\widehat{f * g}(0) = \int_{-\infty}^{\infty} (f * g)(x)\,dx \tag{8.41}$$

Then (8.38) follows from the convolution theorem. □

Theorem 8.7. *Let f and g are two nonnegative functions. Suppose*

$$f(x) > 0 \text{ for all } |x| < a \tag{8.42}$$

$$f(x) = 0 \text{ for all } |x| > a \tag{8.43}$$

$$g(x) > 0 \text{ for all } |x| < b \tag{8.44}$$

$$g(x) = 0 \text{ for all } |x| > b \tag{8.45}$$

Then

$$(f * g)(x) > 0 \text{ for all } |x| < a + b \tag{8.46}$$

$$(f * g)(x) = 0 \text{ for all } |x| \geq a + b \tag{8.47}$$

See Exercise 9.8 for a generalization of Theorem 8.7 as well as some connections to additive combinatorics.

Proof. Let's first prove (8.46). By definition of the convolution, $(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\,dy$. By (8.42) and (8.44), the integrand $f(y)g(x - y)$ is positive if $|y| < a$ and $|x - y| < b$, that is, if y belongs to the following intersection:

$$(-a, a) \cap (x - b, x + b). \tag{8.48}$$

If $|x| < a + b$, the intersection is some nondegenerate interval (c, d) (draw a picture!), so

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x - y)\,dy \geq \int_{c}^{d} f(y)g(x - y)\,dy > 0, \tag{8.49}$$

which completes the proof of (8.46).

To prove (8.47), we will use (8.43) and (8.45). I'll leave the details as an exercise. □

8.5.4 Back to Borwein integrals

For $k \geq 1$, we showed

$$\int_{-\infty}^{\infty} \operatorname{sinc}(\pi t)\, \operatorname{sinc}(\tfrac{\pi t}{3})\, \operatorname{sinc}(\tfrac{\pi t}{5}) \cdots \operatorname{sinc}(\tfrac{\pi t}{13})\, dt = \int_{-1/2}^{1/2} (f_3 * f_5 * \cdots * f_{2k+1})(x)\, dx \tag{8.50}$$

where

$$f_3(x) = 3 \cdot \mathbb{1}_{[-\frac{1}{6}, \frac{1}{6}]}(x) \tag{8.51}$$

$$f_5(x) = 5 \cdot \mathbb{1}_{[-\frac{1}{10}, \frac{1}{10}]}(x) \tag{8.52}$$

$$f_7(x) = 7 \cdot \mathbb{1}_{[-\frac{1}{14}, \frac{1}{14}]}(x) \tag{8.53}$$

and so on.

Note that $\int_{-\infty}^{\infty} f_m(x)\, dx = 1$ for all m. By applying Theorem 8.6 repeatedly, we get

$$\int_{-\infty}^{\infty} (f_3 * f_5 * \cdots * f_{2k+1})(x)\, dx = 1. \tag{8.54}$$

By applying Theorem 8.7 repeatedly, we get that

$$(f_3 * f_5 * \cdots * f_{2k+1})(x) \begin{cases} > 0 & \text{if } |x| < \frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k+1}\right) \\ = 0 & \text{if } |x| > \frac{1}{2}\left(\frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k+1}\right) \end{cases}. \tag{8.55}$$

By combining (8.54) and (8.55), we see that the RHS of (8.50) is

$$\begin{cases} = 1 & \text{if } \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k+1} \leq 1 \\ < 1 & \text{if } \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k+1} > 1 \end{cases}. \tag{8.56}$$

8.6 Application: Heat equation on the line

(We didn't cover this section in class.)

Suppose we have a straight metal rod of infinite length. We can think of points on this rod as elements of $\mathbb{R}$. Suppose the initial temperature is at position x is $f(x)$ for some function $f : \mathbb{R} \to \mathbb{C}$. We'd like to determine the temperature at future times. (See Section 6.1 for more background information about the heat equation.)

Theorem 8.8 (Solution to heat equation)**.** *Suppose $u(x,t)$ satisfies the heat equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$ and the initial condition $u(x,0) = f(x)$. Then (under some mild smoothness conditions on f),*

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{f}(\xi) e^{-4\pi^2\xi^2 t} e^{2\pi i x\xi}\, d\xi \tag{8.57}$$

Proof. You are asked to do this in Exercise 9.9. □

Fun fact 8.9. Define

$$H_t(x) = \int_{-\infty}^{\infty} e^{-4\pi^2\xi^2 t} e^{2\pi i x\xi}\, d\xi. \tag{8.58}$$

Then by the convolution theorem (Theorem 8.5), (8.57) could also be written

$$u(x,t) = (H_t * f)(x). \tag{8.59}$$

The function $H_t(x)$ is called the *heat kernel* (on the line). It is true that

$$H_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t}. \tag{8.60}$$

This follows from the fact that the Fourier transform of the Gaussian $e^{-\pi x^2}$ is itself $e^{-\pi\xi^2}$. See Stein and Shakarchi for the proof.

9 Day 4 exercises

Exercise 9.1. () Fix $\xi_0 \in \mathbb{R}$ and let $f(x) = \mathbb{1}_{[-r,r]}(x)e^{2\pi i x\xi_0}$. Show that

$$\widehat{f}(\xi) = \frac{\sin 2\pi r(\xi - \xi_0)}{\pi(\xi - \xi_0)}. \tag{9.1}$$

Exercise 9.2. () Let $f \in \mathcal{S}(\mathbb{R})$, and let $g = \widehat{f}$. Show that $\widehat{g}(x) = f(-x)$. (In other words, if you take the Fourier transform of $f(x)$ twice, you get back $f(-x)$.)

Exercise 9.3. () Let $f(x) = \frac{\sin \pi\xi}{\pi\xi}$. Show that $\widehat{f}(\xi) = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(\xi)$. (Hint: Don't calculate $\widehat{f}(\xi)$ directly using the definition of the Fourier transform. That's hard!)

Exercise 9.4. () Show that $\int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 dx = 1$ and $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \pi$.

Exercise 9.5. () The Dirac delta function $\delta(x)$ is defined to be a "function" which satisfies the following "mythical" property:

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0) \qquad \text{for all continuous functions } f : \mathbb{R} \to \mathbb{C}. \tag{9.2}$$

Here are some basic calculations you can do with the delta function.

1. Calculate the Fourier transform $\widehat{\delta}(\xi)$.
2. Let $\delta_{x_0}(x) = \delta(x - x_0)$. Calculate the Fourier transform $\widehat{\delta_{x_0}}(\xi)$. (Hint: do a change of variables.)

3. Show that the Fourier transform of $f(x) = e^{2\pi i \xi_0 x}$ is $\widehat{f}(\xi) = \delta_{x_0}(\xi)$.

4. Let f be a continuous function. Show that $\delta * f = f$.

These properties of δ are very important in signal processing.

Don't worry about the details. These calculations are completely rigorous in the framework of distributions. See Fun fact 8.1 for some more details.

Exercise 9.6. ( ) Verify (8.47) of Theorem 8.7.

Exercise 9.7. ( ) Let $b_1, b_2, \cdots, b_N$ be a finite sequence of positive numbers with $b_1 = \max(b_1, b_2, \ldots, b_N)$. Show that

$$\int_{-\infty}^{\infty} \operatorname{sinc}(b_1 t) \cdots \operatorname{sinc}(b_N t)\, dt \begin{cases} = \frac{\pi}{b_1} & \text{if } \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_N} \leq \frac{1}{b_1} \\ < \frac{\pi}{b_1} & \text{if } \frac{1}{b_2} + \frac{1}{b_3} + \cdots + \frac{1}{b_N} > \frac{1}{b_1} \end{cases}. \tag{9.3}$$

Exercise 9.8. ( ) Here we prove a generalization of Theorem 8.7. First, if $A, B \subset \mathbb{R}$, we define the *Minkowski sum* by

$$A + B = \{a + b : a \in A, b \in B\}. \tag{9.4}$$

Let A and B be two open sets in $\mathbb{R}$. Let f and g are two nonnegative functions. Suppose

$$f(x) > 0 \text{ for all } x \in A \tag{9.5}$$

$$f(x) = 0 \text{ for all } x \notin A \tag{9.6}$$

$$g(x) > 0 \text{ for all } x \in B \tag{9.7}$$

$$g(x) = 0 \text{ for all } x \notin B \tag{9.8}$$

1. Show that $(f * g)(x) > 0$ for all $x \in A + B$.

2. Show that $(f * g)(x) = 0$ for all $x \notin A + B$.

Fun fact: The Minkowski sum appeared in Milan's week 2 class on the Plünnecke-Ruzsa inequality, and is a key concept in the field of additive combinatorics. It turns out that Fourier analytic methods are very useful in this field.

Exercise 9.9. ( ) Give a fishy ( ) proof of Theorem 8.8. Hint: Take a look at the proof of Theorem 6.1 and replace (6.3) with the following:

$$u(x,t) = \int_{-\infty}^{\infty} \widehat{u}(\xi, t) e^{2\pi i x \xi}\, d\xi \qquad \text{where } \widehat{u}(\xi, t) = \int_{-\infty}^{\infty} u(x,t) e^{-2\pi i x \xi}\, dx. \tag{9.9}$$

Now you should be able to imitate the proof of Theorem 6.1 very closely.

10 Day 5

10.1 Time-frequency analysis

The following is all a very informal discussion.

Let's say you have a signal $f(t)$, where t represents time. And suppose that we want to find a function $\phi(t, \xi)$ which measures the amplitude/strength of frequency ξ at time t in the signal.

One way to visualize $\phi(t, \xi)$ is as a heat map of the *phase space* $\{(t, \xi) : t \in \mathbb{R}, \xi \in \mathbb{R}\}$. We can darken points (t, ξ) of the plane, if frequency ξ is strong at time t. We just invented the musical score! As well as the spectrogram! (Speaking of spectrograms, Audacity has a feature that let's you view your sound file as a spectrogram.)

Suppose we wanted $\phi(0, \xi)$. Here's how we could do this. We could "localize in time" by considering the product $\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)f(t)$. Then we can compute the Fourier transform of this product:

$$\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}f}(\xi) = \int_{-\infty}^{\infty} \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)f(t)e^{-2\pi it\xi}\,dt. = \int_{-1/2}^{1/2} f(t)e^{-2\pi it\xi}\,dt. \tag{10.1}$$

(What we're describing here is called the *short-time Fourier transform*. See https://en.wikipedia.org/wiki/Short-time_Fourier_transform.)

However, one issue with this approach is that it's very hard to tell apart frequencies that are close. For example, suppose you have a signal $f(t)$ and you look at just $t \in [-\frac{1}{2}, \frac{1}{2}]$, and that you see something that is approximately a horizontal line. Is this a signal of frequency 0? Or maybe a signal of frequency 0.001? If the frequency were 0.001, we would need a much bigger time interval than $[-\frac{1}{2}, \frac{1}{2}]$ to be able to differentiate it from frequency 0. For example, it would take an interval like $[-500, 500]$ to be able to see an entire period of the signal of frequency 0.001.

Similarly, to differentiate between a frequency of ξ_0 and a frequency of $\xi_0 + 0.001$, we would need an interval like $[-500, 500]$ or bigger.

The following example illustrates the ideas above with some calculations.

Example 10.1. Consider $f(t) = c_0 e^{2\pi it\xi_0}$ which is a signal purely of frequency ξ_0, with amplitude c_0. Then (10.1) becomes

$$\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}f}(\xi_0) = \int_{-1/2}^{1/2} c_0 e^{2\pi it\xi_0}e^{-2\pi it\xi_0}\,dt. = \int_{-1/2}^{1/2} c_0\,dt = c_0 \tag{10.2}$$

as expected. If $\xi \neq \xi_0$, then we should expect that there is no strength of frequency ξ at time 0. However, the calculations (Exercise 9.1) tell us that

$$\widehat{\mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}f}(\xi_0) = c_0 \operatorname{sinc}(\pi(\xi - \xi_0)), \tag{10.3}$$

which is not zero! Recall that $\operatorname{sinc} x = \frac{\sin x}{x}$, so $\lim_{|x|\to\infty} \operatorname{sinc} x = 0$. This implies that when $|\xi - \xi_0|$ is very large, the the RHS of (10.3) is very small, which is good. But when $|\xi - \xi_0|$, then the RHS is close to c_0. This is because we cannot tell close frequencies apart.

Now suppose that we "localize" to a different time interval $[-\frac{r}{2}, \frac{r}{2}]$. Observe that

$$\frac{1}{r}\widehat{\mathbb{1}_{[-\frac{r}{2},\frac{r}{2}]}f}(\xi_0) = \frac{1}{r}\int_{-r/2}^{r/2} c_0 e^{2\pi i t\xi_0} e^{-2\pi i t\xi_0}\,dt. = \frac{1}{r}\int_{-r/2}^{r/2} c_0\,dt = c_0 \tag{10.4}$$

as expected. For $\xi \neq \xi_0$, we get

$$\frac{1}{r}\widehat{\mathbb{1}_{[-\frac{r}{2},\frac{r}{2}]}f}(\xi_0) = c_0 \operatorname{sinc}(\pi r(\xi - \xi_0)), \tag{10.5}$$

Compared to (10.3), the RHS of (10.5) has been squeezed horizontally by a factor of r. If r is much greater than 1, then we see $|\xi - \xi_0|$ needs to be really small for the RHS of (10.5) to be close to c_0.

Thanks to the extra factor of r in (10.5), we can say that the "frequency resolution" has improved by a factor of r. However, the cost is that the "time resolution" has worsed by the same factor of r: our time interval is now r times bigger. When we measure the strength of a frequency this way, we don't know where in $[-\frac{r}{2}, \frac{r}{2}]$ the frequency is strong.

In general, if we try to increase (i.e., improve) the frequency resolution, the time resolution decreases (worsens) by the same factor. Conversely, if we try to increase the time resolution, the frequency resolution decreases by the same factor. This means that it is impossible to define a function $\phi(t, \xi)$ to measure the strength of frequency ξ (exactly) at time t (exactly). We instead have

$$(\text{uncertainty in time}) \cdot (\text{uncertainty in frequency}) \geq \text{constant} \tag{10.6}$$

This is a manifestation of the *Heisenberg uncertainty principle*. This principle comes in many forms, but they all have something to do with the impossibility of "localizing" both f and $\widehat{f}$. (In quantum mechanics, f corresponds to the "position operator" and $\widehat{f}$ corresponds to the "momentum operator.") See Neeraja's Week 4 class for more information about the Heisenberg uncertainty principle.

Fun fact 10.2. The usual formulation of the uncertainty principle is the following:

$$\text{If } \int_{-\infty}^{\infty} |f(x)|^2\,dx = 1, \quad \text{then } \left(\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{f}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}. \tag{10.7}$$

Fun fact 10.3. If we want to avoid issues caused by the uncertainty principle in our study of the phase space above, we could use wavelets. https://en.wikipedia.org/wiki/Wavelet

10.2 Some other things you can do with the Fourier transform

The following is true for all functions $f \in \mathcal{S}(\mathbb{R})$ and is called the *Poisson summation formula*:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \tag{10.8}$$

Unlike most things we've seen about the Fourier transform, this has no analogue for Fourier series. To give a demonstration of how useful it is, the Poisson summation formula can be used to calculate the series $\sum_{n=1}^{\infty} \frac{1}{n^{2k}}$, prove quadratic reciprocity, the functional equation for the zeta function, the Whittaker–Shannon interpolation formula (Exercise 11.1) from information theory, and more.

10.3 Fourier analysis on $\mathbb{Z}/N\mathbb{Z}$

With Fourier series, we had $f : \mathbb{R} \to \mathbb{C}$ a 1-periodic function. But now suppose we replace $\mathbb{R}$ with $\frac{1}{N}\mathbb{Z} = \{\frac{k}{N} : k \in \mathbb{Z}\}$. (Warning: the group $\frac{1}{N}\mathbb{Z}$ is not the same as $\mathbb{Z}/N$, which is the group of integers modulo N.)

So now suppose $f : \frac{1}{N}\mathbb{Z} \to \mathbb{C}$ is a 1-periodic function. How can we define Fourier series for f? Before we had an integral. But now we only have a few values that we can plug into f. So let's change the integral into a Riemann sum! The integral was $\int_0^1 f(x)e^{-2\pi inx}\,dx$ (See (8.1).) Now we split $[0,1]$ into N intervals to get the following Riemann sum:

$$a_n = \frac{1}{N}\sum_{k=0}^{N-1} f\left(\tfrac{k}{N}\right) e^{-2\pi ink/N} \qquad n = 0, 1, \ldots, N-1 \tag{10.9}$$

Then we do indeed have the reconstruction/inversion formula (analogue of (8.2))

$$f(x) = \sum_{n=0}^{N-1} a_n e^{2\pi inx}, \qquad x = 0, \frac{1}{N}, \ldots, \frac{N-1}{N} \tag{10.10}$$

as well as the Parseval-Plancherel formula (analogue of (8.3))

$$\frac{1}{N}\sum_{k=0}^{N-1} \left|f\left(\tfrac{k}{N}\right)\right|^2 = \sum_{n=0}^{N-1} |a_n|^2 \tag{10.11}$$

We can think of the function f above as being defined on the group $\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\}$ with addition modulo 1. But perhaps it's nicer to rescale everything by N, to get the group $\{0, 1, \ldots, N-1\}$ with addition modulo N. This group is denoted $\mathbb{Z}/N\mathbb{Z}$ or $\mathbb{Z}/N$. Let's restate the results above in terms of this group.

Theorem 10.4. *Let $F : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ be any function. Define*

$$a_n = \frac{1}{N}\sum_{k=0}^{N-1} F(k)e^{-2\pi ink/N} \qquad n = 0, 1, \ldots, N-1 \tag{10.12}$$

Then

$$F(k) = \sum_{k=0}^{N-1} a_n e^{2\pi ink/N}, \qquad k = 0, 1, \ldots, N-1 \tag{10.13}$$

and

$$\frac{1}{N}\sum_{k=0}^{N-1} |F(k)|^2 = \sum_{n=0}^{N-1} |a_n|^2 \tag{10.14}$$

Proof. We actually didn't give the proofs in the previous settings (when we considered $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{C}$ or $f : \mathbb{R} \to \mathbb{R}$), because we had to worry about infinite sums and improper integrals. But in the current setting, there are no integrals and all the sums are finite! So we can actually give a proof!! Except by "we," I mean "you." See Exercise 11.2. □

Fun fact 10.5. In this section, "a wave of frequency α" means $e^{2\pi i\alpha x} = \cos 2\pi\alpha x + i\sin 2\pi\alpha x$.

If we sample a signal $f : [0, 1] \to \infty$ at times spaced by $\frac{1}{N}$, then a wave of frequency N looks like a constant wave. Mathematically, $e^{2\pi i(\alpha+N)x} = e^{2\pi i\alpha x}$ for all $x \in \frac{1}{N}\mathbb{Z}$.

In general, we cannot tell the difference between a wave of frequency α and a wave of frequency $\alpha + N$. Mathematically, $e^{2\pi i(\alpha+N)x} = e^{2\pi i\alpha x}$ for all $x \in \frac{1}{N}\mathbb{Z}$.

So a wave of frequency $\alpha = \frac{3}{4}N$ looks like a wave of frequency $\alpha = -\frac{1}{4}N$. However, if we knew in advance that our signal only had frequencies between $-\frac{1}{2}N$ and $\frac{1}{2}N$, then there's no ambiguity.

This is related to the Nyquist–Shannon sampling theorem and the Whittaker–Shannon interpolation formula.

Fun fact 10.6. If we have a function $F : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ and we'd like to calculate the Fourier coefficients $(a_n)_{n=0}N - 1$, using (10.12) directly would result in $O(N^2)$ multiplications, which is too slow for many practical applications. Fortunately, there's a much better way, called the *fast Fourier transform*, which allows us to calculate the Fourier coefficients in only $O(N \log N)$ multiplications. It is thanks to this algorithm that modern signal processing works!

See Figure 1 for a comparison of the three kinds of Fourier transforms we have encountered so far.

setting	functions on $[0,1]$	functions on $\mathbb{R}$	discrete functions
domain of $f(x)$	$\mathbb{R}/\mathbb{Z}$	$\mathbb{R}$	$\mathbb{Z}/N\mathbb{Z}$
domain of $\widehat{f}(\xi)$	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{Z}/N\mathbb{Z}$
F. transform $\widehat{f}(\xi)$	$\int_{\mathbb{R}/\mathbb{Z}} f(x)e^{-2\pi ix\xi}\,dx$	$\int_{\mathbb{R}} f(x)e^{-2\pi ix\xi}\,dx$	$\frac{1}{N}\sum_{x\in\mathbb{Z}/N\mathbb{Z}} f(x)e^{-2\pi ix\xi/N}$
F. inversion $f(x)$	$\sum_{\xi\in\mathbb{Z}} \widehat{f}(\xi)e^{2\pi ix\xi}$	$\int_{\mathbb{R}} \widehat{f}(\xi)e^{2\pi ix\xi}\,d\xi$	$\sum_{\xi\in\mathbb{Z}/N\mathbb{Z}} \widehat{f}(\xi)e^{2\pi ix\xi/N}$
Parseval/Plancherel	$\int_{\mathbb{R}/\mathbb{Z}} \lvert f(x)\rvert^2\,dx$ $= \sum_{\xi\in\mathbb{Z}} \lvert\widehat{f}(\xi)\rvert^2$	$\int_{\mathbb{R}} \lvert f(x)\rvert^2\,dx$ $= \int_{\mathbb{R}} \lvert\widehat{f}(\xi)\rvert^2\,d\xi$	$\frac{1}{N}\sum_{x\in\mathbb{Z}/N\mathbb{Z}} \lvert f(x)\rvert^2$ $= \sum_{\xi\in\mathbb{Z}/N\mathbb{Z}} \lvert\widehat{f}(\xi)\rvert^2$

Figure 1: Some connections among the Fourier transforms of various settings. Note that $\int_{\mathbb{R}/\mathbb{Z}}$ just means $\int_0^1$.

10.4 The Fourier transform in higher dimensions

There are higher-dimensional versions of the Fourier transform. To go from 1-dimensional to d-dimensional, just make both x and ξ vectors (with d components) and replace $x\xi$ in the formulas with the dot product $x \cdot \xi$. See Figure 2 for a summary.

Fun fact 10.7. If we consider Fourier analysis on $(\mathbb{Z}/2\mathbb{Z})^d$ (see Figure 2), then this is precisely the setting encountered in Tim!'s Week 1 Fourier analysis of Boolean functions class.

Fun fact 10.8. In $\mathbb{R}^d$ or $(\mathbb{R}/\mathbb{Z})^d$, there are analogues of low-pass filters that we talked about. However, we have to be careful. Some low-pass filters behave very badly. I unfortunately cannot talk about this in more detail without first introducing a lot of background. But I will say that the reason such filters can behave badly in higher dimensions is because of the existence of Kakeya sets, a topic in my Week 4 class.

10.5 The Fourier transform in even more generality

Here is a very general statement that contains all the results in the tables above as special cases. To understand the statement, you need to know some group theory, some topology,

setting	functions on $[0,1]^d$	functions on $\mathbb{R}^d$	discrete functions
domain of $f(x)$	$(\mathbb{R}/\mathbb{Z})^d$	$\mathbb{R}^d$	$(\mathbb{Z}/N\mathbb{Z})^d$
domain of $\widehat{f}(\xi)$	$\mathbb{Z}^d$	$\mathbb{R}^d$	$(\mathbb{Z}/N\mathbb{Z})^d$
F. transform $\widehat{f}(\xi)$	$\displaystyle\int_{(\mathbb{R}/\mathbb{Z})^d} f(x)e^{-2\pi i x\xi}\,dx$	$\displaystyle\int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}\,dx$	$\displaystyle\frac{1}{N^d}\sum_{x\in(\mathbb{Z}/N\mathbb{Z})^d} f(x)e^{-2\pi i x\xi/N}$
F. inversion $f(x)$	$\displaystyle\sum_{\xi\in\mathbb{Z}^d}\widehat{f}(\xi)e^{2\pi i x\xi}$	$\displaystyle\int_{\mathbb{R}^d}\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi$	$\displaystyle\sum_{\xi\in(\mathbb{Z}/N\mathbb{Z})^d}\widehat{f}(\xi)e^{2\pi i x\xi/N}$
Parseval/Plancherel	$\displaystyle\int_{(\mathbb{R}/\mathbb{Z})^d}\lvert f(x)\rvert^2\,dx$ $\displaystyle=\sum_{\xi\in\mathbb{Z}^d}\lvert\widehat{f}(\xi)\rvert^2$	$\displaystyle\int_{\mathbb{R}^d}\lvert f(x)\rvert^2\,dx$ $\displaystyle=\int_{\mathbb{R}^d}\lvert\widehat{f}(\xi)\rvert^2\,d\xi$	$\displaystyle\frac{1}{N^d}\sum_{x\in\mathbb{Z}/N\mathbb{Z}}\lvert f(x)\rvert^2$ $\displaystyle=\sum_{\xi\in(\mathbb{Z}/N\mathbb{Z})^d}\lvert\widehat{f}(\xi)\rvert^2$

Figure 2: Higher dimensional version of Figure 1. Note that $dx = dx_1\,dx_2\,\cdots\,dx_d$ and similarly for $d\xi$.

and some measure theory. See https://en.wikipedia.org/wiki/Pontryagin_duality for more information.

Theorem 10.9. *Let G be a locally compact abelian group with Haar measure μ. Let $\widehat{G}$ be the Pontryagin dual of G and let ν be the dual measure. For $f \in L^1(G)$, define the Fourier transform of f by*

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}\,d\mu(x). \tag{10.15}$$

If $\widehat{f} \in L^1(\widehat{G})$, then we have the Fourier inversion formula

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(x)\,d\nu(\chi). \tag{10.16}$$

If $f \in L^1(G) \cap L^2(G)$, then we have the Parseval/Plancherel theorem

$$\int_G |f(x)|^2\,d\mu(x) = \int_{\widehat{G}} |\widehat{f}(\chi)|^2\,d\nu(\chi). \tag{10.17}$$

Fun fact 10.10. All finite abelian groups are compact and hence locally compact, so we can do Fourier analysis on them. In fact, Fourier analysis on finite abelian groups is closely related to the representation theory of these groups.

11 Day 5 exercises

Exercise 11.1. ( ) Suppose that $f \in \mathcal{S}(\mathbb{R})$ is a function such that $\widehat{f}(\xi) = 0$ for all $|\xi| > \frac{1}{2}$. Prove that

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x-n)}{\pi(x-n)} \tag{11.1}$$

This formula says that if f only has small frequencies (smaller than $\frac{1}{2}$), and we know the values of f at the integers, then we can perfectly reconstruct the whole function $f(x)$. Hint: Use the Poisson summation formula (10.8).

This is known as the *Whittaker–Shannon interpolation formula.*

Exercise 11.2. ( ) Prove Theorem 10.4. There are several ways you could do this. You could use define an inner product as in Fun fact 2.6. Or you could just expand out all the sums and rearrange terms around until everything cancels.