Introduction to Analysis

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Mathcamp 2020 Week 1

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1 Introduction

1.1 Course blurb

This class is a rigorous introduction to limits and related concepts in calculus. Consider the following questions:

1. Every calculus student knows that $\frac{d}{dx}(f+g) = f' + g'$. Is it also true that $\frac{d}{dx}\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} f_n'$?
2. Every calculus student knows that $a + b = b + a$. Is it also true that you can rearrange terms in an infinite series without changing its sum?

Sometimes, things are not as they seem. For example, the answer to the second question is a resounding "no." The Riemann rearrangement theorem, which we will prove, states that we can rearrange the terms in infinite series such as $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ so that the sum converges to π, e, or whatever we want!

To help us study the questions above and many other ones, the key tool we'll use is the "epsilon-delta definition" of a limit. This concept can be hard to work with at first, so we will study many examples and look at related notions, such as uniform convergence. Being comfortable reasoning with limits is central to the field of mathematical analysis, and will open the door to some very exciting mathematics.

1.2 About this class

This class is a rigorous introduction to limits and related concepts in calculus. The key concept is the formal definition of a limit, sometimes called the "epsilon-delta definition." The definition itself is not long and not hard to memorize. But working with this definition can be difficult. So that is the main focus of this class, and the main thing I would like you to gain from this class, is the *technique* of working with limits and related concepts. The theorems we prove are not as important as *how* we prove them. Nevertheless, we will see some beautiful and some surprising theorems, and being proficient with epsilon-delta proofs will open the door to some very exciting areas of mathematics.

1.3 About these notes

These notes will be updated after class each day to reflect what was covered and to provide exercises for that day. Note that these notes are very rough, and probably don't make much sense unless you have been to lecture. In particular, pictures are very important for this class, but there are no pictures here!

1.4 Textbook references

A good introductory text on this topic is Michael Spivak's *Calculus* (published by Publish or Perish). Chapters 5, 6, 22, 23, 24 are particularly relevant.

2 Day 1

2.1 Why we need rigor in calculus

Here I will present some phenomena that arise when we study infinite series. Recall that infinite series are defined by taking a limit of the partial sums: $\sum_{k=1}^{\infty} a_k$ is defined by $\lim_{n\to\infty} \sum_{k=1}^{n} a_k$. This means that limits are lurking in the background.

2.1.1 Infinite series of functions

Suppose f and g are two functions defined on $[0,1]$. We know that addition has the following properties.

1. If f and g are continuous, then $f+g$ is continuous.
2. If f and g are differentiable, then $(f+g)' = f' + g'$.
3. If f and g are Riemann integrable, then $\int_0^1 (f(x)+g(x))\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx$.

These results also hold for finite sums of functions. But what about infinite sums? Suppose we have a sequence of functions $f_1, f_2, \ldots$ defined on $[0,1]$. Let $g(x) = \sum_{k=1}^{\infty} f_k(x)$.

1. If the f_n are all continuous, is it true that g is continuous?

2. If the f_n are all differentiable, is it true that $g'(x) = \sum_{n=1}^{\infty} f_n'(x)$?

3. If the f_n are all Riemann integrable, is it true that $\int_0^1 g(x)\,dx = \sum_{n=1}^{\infty} \int_0^1 f_n(x)\,dx$?

The answer to all three is "not in general." It turns out that in some situations, the results are true. For example, for Taylor series, you can differentiate and integrate term by term with no problem.

In general, what assumptions about the sequence f_n should we make to guarantee that these statements continue to hold for infinite sums?

2.1.2 Rearranging terms in a series

In calculus class, you learned that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2. \tag{2.1}$$

(Note that mathematicians typically use log to denote the natural log.)

What happens if we decide to sum up the terms in a different order? We could rearrange the terms in the infinite series to obtain:

$$\begin{aligned} &\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots & (2.2)\\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots & (2.3)\\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) & (2.4)\\ &= \frac{1}{2}\log 2 & (2.5)\end{aligned}$$

For finite sums, we can rearrange terms with no issue. This leads to some natural questions:

1. When can we rearrange terms in infinite sums without changing the value?

2. When the values do change, which values could we actually achieve?

2.2 Definition of a limit

(Reference: Spivak, Chapter 5.)

The limit is one of the most important and fundamental concepts in calculus.

The equation $\lim_{x\to c} f(x) = L$ means, roughly speaking that as x approaches c, $f(x)$ gets close to L. But what is meant by "approaches" and "gets close to"?

The concept was not formalized until the 1800s.

Here is the formal definition of a limit:

Definition 2.1. $\lim_{x \to c} f(x) = L$ means: "For all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$."

2.3 Quantifiers

We'll use the following symbols: $\forall$ means "for all," and $\exists$ means "there exists." Also, "s.t." stands for "such that."

Example 2.2. Consider the following statements.

1. $\forall x \in \mathbb{R}, x^2 \geq 0$. True.
2. $\forall x \in \mathbb{R}, x^2 \geq 1$. False.
3. $\forall x \in \mathbb{R}, x^2 \leq -1$. False.
4. $\exists x \in \mathbb{R}$ s.t. $x^2 \geq 0$. True.
5. $\exists x \in \mathbb{R}$ s.t. $x^2 \geq 1$. True.
6. $\exists x \in \mathbb{R}$ s.t. $x^2 \leq -1$. False.

So far so good. But things can become more complicated when we string many of the quantifiers together.

Example 2.3. Consider the following statements.

1. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x + y = 0$. True.
2. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = 1$. False.
3. $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}, x + y = 0$. False

Definition 2.4. $\lim_{x \to c} f(x) = L$ means:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, \text{ if } 0 < |x - c| < \delta \text{ then } |f(x) - L| < \varepsilon \tag{2.6}$$

Yikes. Let's just look at the last part: $\forall x,$ if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

We can think of $0 < |x - c| < \delta$ as the vertical strip between $x = c \pm \delta$. We also have to remove the line $x = c$.

We can think of $|f(x) - L| < \varepsilon$ as the horizontal strip between $y = f(x) \pm L$.

A geometric interpretation: The statement means that whenever we're at a point on the graph of f inside the vertical strip, then it also lies inside the horizontal strip.

Example 2.5. Consider $f(x) = x^2, c = 2, L = 4$.

1. $\forall x,$ if $0 < |x-2| < 1,$ then $|x^2-4| < 5$. True. (If you replace 5 with something smaller, it is not true. If you replace the 1 with something larger, it is not true.)

Protip 2.6. Whenever you see the expression $|a-b|$, you can mentally think of this as quantity as "the distance between a and b." That will help you think about the limit definition geometrically.

2.4 The definition of a limit as a game (or Pokémon® Battle)

Start with a function $f(x)$ and two real numbers c and L.

Let's break the limit definition down into four parts:

1. $\forall \varepsilon > 0,$
2. $\exists \delta > 0$ s.t.
3. $\forall x,$ if $0 < |x-c| < \delta$ then
4. $|f(x) - L| < \varepsilon$

We can take these four parts and translate them into a "game" between two players, who we will call Delphox and Espeon. Delphox want to show $\lim_{x\to c} f(x) = L$, and Espeon wants to show this is false.

Here are the steps of the game:

1. Espeon starts by choosing $\varepsilon > 0$.
2. Delphox responds by choosing $\delta > 0$.
3. Espeon responds by choosing x satisfying $0 < |x-c| < \delta$.
4. Delphox responds by proving that $|f(x) - L| < \varepsilon$.

If Espeon can play the game in a way to make Delphox unable to complete the last step, then Espeon wins. Otherwise, Delphox wins.

Protip 2.7. Note that each step of the game corresponds to a part of the statement of the limit definition.

In general, if you see a mathematical statement with lots of $\forall$s and $\exists$s, you can convert it into a two-player game, in the same way as we did above.

2.4.1 A more geometric description

Start by drawing the graph $y = f(x)$.

Step 1 of the game: Espeon starts by choosing some $\varepsilon > 0$. E draws the horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ on the graph.

Step 2 of the game: Delphox has to respond by choosing some $\delta > 0$. D draws the vertical lines $x = c - \varepsilon$ and $x = c + \varepsilon$ on the graph.

Steps 3 and 4 of the game: Delphox's response is valid if the following is satisfied: In between Delphox's two vertical lines, the graph of f lies entirely within Espeon's horizontal lines (with the possible exception of the point $(c, f(c))$).

There are two possibilities:

If Espeon has a move for which Delphox cannot respond, then Espeon wins, and $\lim_{x\to c} f(x) \neq L$.

If no matter how Espeon moves, Delphox always has a response, then Delphox wins, and $\lim_{x\to c} f(x) = L$.

2.5 Examples

We'll see some examples of proving statements of the form $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) \neq L$. Many textbooks will make a statement and jump straight to its proof. It might be tempting to do this in real life, but there's usually some kind of thought process/scratch-work involved before we can write down the proof. In the examples that follow, I'll explain some scratch-work that we can do to bring us to the proof.

Example 2.8. Let's show $\lim_{x\to 3} 2x = 6$.

Scratch-work. Espeon starts with by choosing $\varepsilon > 0$. How can Delphox respond? By choosing $\delta = \frac{1}{2}\varepsilon$. We can see from a picture that this clearly works.

Proof. Suppose $\varepsilon > 0$. Then let $\delta = \frac{1}{2}\varepsilon$. Suppose $0 < |x - 3| < \delta$. Then $|2x - 6| = 2|x - 3| < 2\delta = \varepsilon$. □

Protip 2.9. In the example above, we don't have to take $\delta = \frac{1}{2}\varepsilon$. For example, $\delta = \frac{1}{10}\varepsilon$ would also work. We can take anything smaller than $\frac{1}{2}\varepsilon$. In general, there is no single correct value of δ.

Protip 2.10. Every proof of $\lim_{x\to c} f(x) = L$ more or less has the following structure.

1. Suppose $\varepsilon > 0$.

2. Then let $\delta =$ `[some expression involving epsilon]`

3. Suppose $0 < |x - c| < \delta$.

4. Then `[some argument to show that` $|f(x) - L| < \varepsilon$`]`

This structure is dictated by the definition of the limit. It also follows the structure of the game between Espeon and Delphox described above.

Example 2.11. Consider the statement $\lim_{x\to 3} 2x \neq 0$.

Scratch-work. Since we do not want to show inequality, we should imagine ourselves as Delphox. What can we choose so as ε so that Espeon has no valid response? Let's try $\varepsilon = 1$. If Delphox tries to respond with some $\delta > 0$, we can show that the number $3 + \frac{\delta}{2}$ lies between the two vertical lines, but $2(3 + \frac{\delta}{2})$ does not lie between the two horizontal lines.

Proof. Let $\varepsilon = 1$. Suppose $\delta > 0$. Then let $x = 3 + \frac{\delta}{2}$. Then $0 < |x - 3| < \delta$, but $|2x - 0| = 6 + \delta > 1 = \varepsilon$. □

Protip 2.12. Every proof of $\lim_{x\to c} f(x) \neq L$ more or less has the following structure.

1. Let $\varepsilon =$ `[something]`

2. Suppose $\delta > 0$

3. Let $x =$ `[something which satisfies` $0 < |x - c| < \delta$ `]`

4. Then $0 < |x - c| < \delta$ and `[some argument to show that` $|f(x) - L| \geq \varepsilon$`]`

This structure is dictated by the definition of the limit.

Example 2.13. Consider $\lim_{x\to 2} x^2 = 4$.

Scratch-work. For the linear functions above, it was easy to consider all ε at once. Here, it's not as easy. Let's think about a single ε as an example. Suppose Espeon chooses $\varepsilon = 1$. We need to find δ. We can consider $|x^2 - 4| < 1$. This means $3 < x^2 < 5$, so $\sqrt{3} < |x| < \sqrt{5}$. So we can take $\delta = \min(\sqrt{5} - 2, 2 - \sqrt{3}) = \sqrt{5} - 2 \approx 0.236$.

By the same reasoning, for general ε, we should be able to take $\delta = \sqrt{4 + \varepsilon} - 2$.

Proof. I'll leave the details to you as an exercise. See Exercise 3.4. □

The previous example worked because it's easy to solve for x in $|x^2 - 4| < \varepsilon$. However, if your function $f(x)$ is a more complicated polynomial than x^2, it may be very difficult to solve for x in $|f(x) - L| < \varepsilon$.

Fun fact 2.14. In fact, for a polynomial $P(x)$ of degree 5 or higher, then there does not exist a general formula for the roots of P in terms of radicals. To prove this, you need abstract algebra, in particular, Galois theory.

Let's see how to handle x^2 in a different way below. It may seem more complicated, but the approach is more flexible.

Example 2.15. Consider again $\lim_{x\to 2} x^2 = 4$.

Scratch-work. If B chooses ε. We want to make sure that if $0 < |x-2| < \delta$, then $|x^2-4| < \varepsilon$.

$|x^2 - 4| = |x-2||x+2| < \delta|x+2|$. We haven't chosen δ yet. But at this point, we can decide that we'll always choose $\delta \leq 1$. So $|x-2| < 1$. This implies $|x+2| < 5$. Draw a picture to see this!

So $|x^2-4| < 5\delta$. So we can decide to always choose $\delta < \frac{1}{5}\varepsilon$. Taking $\delta = \min(\frac{1}{5}\varepsilon, 1)$ works!

Proof. Suppose $\varepsilon > 0$. Then let $\delta = \min(\frac{1}{5}\varepsilon, 1)$. Suppose $0 < |x-2| < \delta$. Then $|x-2| < 1$, so $|x+2| < 5$. Also, $|x^2-4| = |x-2||x+2| < \delta \cdot 5 \leq \frac{1}{5}\varepsilon \cdot 5 = \varepsilon$. □

2.6 Triangle inequality

The triangle inequality, in its most basic form, states that for all real numbers a and b, we have $|a+b| \leq |a| + |b|$. We'll use it a lot in this class.

For example, it tells us $|x-y| = |(x-z)+(z-y)| \leq |x-z| + |y-z|$. (This makes a lot of sense geometrically if you think in terms of "distances.")

In Example 2.15, we used the triangle inequality to deduce $|x+2| = |x-4+2| \leq |x-2| + 4 < 1 + 4 = 5$.

2.7 Limits are unique

In the definition of a limit, I gave a definition for the entire statement "$\lim_{x\to c} f(x) = L$." From this definition, it does not immediately follow that a function cannot have two different limits as $x \to c$. We need to prove it!

Theorem 2.16. *If* $\lim_{x\to c} f(x) = L_1$ *and* $\lim_{x\to c} f(x) = L_2$*, then* $L_1 = L_2$

Scratch-work. First of all, notice that this proof will be different from all the previous proofs. Why? In the previous proofs, we had to prove that a statement like $\lim_{x\to c} f(x) = L_1$ is true. But now we are given that such a statement is true.

Suppose for contradiction that $L_1 \neq L_2$. How can $f(x)$ get close to both L_1 and L_2? Intuitively, it can't!

The statements $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_1$ tell us something about horizontal strips around $y = L_1$ and $y = L_2$. Working with two horizontal strips feels fishy... we should be able to get a contradiction somehow... but how?

Proof. This is for you to figure out! See Exercise 3.12. □

Fun fact 2.17. In point-set topology, there is a definition of a limit that works in more situations and for more spaces. In some of these spaces, the above theorem is false. Limits there are not necessarily unique! Yikes!

2.8 Continuity

Definition 2.18. We say that f is *continuous at* c if $\lim_{x \to c} f(x) = f(c)$. Equivalently:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, \text{ if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \varepsilon \tag{2.7}$$

Note that now there is no reason to exclude $x = c$. In fact, in the examples above, there was no reason to exclude $x = c$. So we were in fact proving statements about continuity.

Definition 2.19. Let $f : A \to \mathbb{R}$. We say that f is *continuous* if for all $c \in A$, f is continuous at c. Equivalently:

$$\forall c \in A, \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, \text{ if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \varepsilon \tag{2.8}$$

3 Day 1 Exercises

3.1 Difficulty scale

Here is my attempt at making a difficulty scale.

-  : Should be fairly routine. Try copying or adapting something from class or from the notes, or try applying some theorem we learned.
-  : Requires a bit more thought.
-  : Requires a solid understanding of the relevant concepts.

Don't be afraid of difficult problems! It's by struggling with these exercises that you really learn.

3.2 Required problems for Day 1

At the very least, you should do (or know how to do) every problem labeled and read the statements of all the other problems. If you have difficulties with any of these, please come find me during TAU or send me a message on Slack at any time (including after TAU).

Exercise 3.12 is a particularly important problem to check if you understand the limit definition. You should make a serious attempt to solve the problem. (This doesn't have to be by yourself! Talk to others or come talk to me!)

3.3 Some exercises with specific functions

For each of these, write down a full proof. You don't need to write down your scratch-work, but the process can be very helpful.

Exercise 3.1. ( ) Show $\lim_{x\to -1} 3 = 3$.

Exercise 3.2. ( ) Show $\lim_{x\to 5}(-2x+4) = -6$.

Exercise 3.3. ( ) Let $a, b, c \in \mathbb{R}$. Show that $\lim_{x\to c}(ax+b) = ac+b$. (Hint: You might want to consider the cases $a = 0$ and $a \neq 0$ separately.)

Exercise 3.4. ( ) Finish the argument in Example 2.13.

Exercise 3.5. ( ) Show $\lim_{x\to 1}(x^2+3x-2) = 2$. (Hint: You can try to adapt the method in Example 2.13 or in Example 2.15. One of these will be easier to use here. Which one?)

Exercise 3.6. ( ) Show $\lim_{x\to 2} x^3 = 8$.

Exercise 3.7. ( ) Let $p(x)$ be any polynomial. (So $p(x)$ is of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.) Let c be any real number. Prove that $\lim_{x\to c} p(x) = p(c)$.

Exercise 3.8. ( ) Show $\lim_{x\to 0} \sin\frac{1}{x} \neq 0$.

Exercise 3.9. ( ) The symbol $\mathbb{Q}$ denotes the set of rational numbers. The *characteristic function of the rationals* is the following function:

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \tag{3.1}$$

Show that $f(x) = 1_{\mathbb{Q}}(x)$ is not continuous at $x = 0$.

Exercise 3.10. ( ) Show that $f(x) = x1_{\mathbb{Q}}(x)$ is continuous at $x = 0$.

3.4 Limits for general functions

Exercise 3.11. ( ) (One sided limits) Without looking them up, figure out what the definitions for $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$ should be. (Hint: Think about the pictures we described in class, and how these should the modified to account for one-sided limits.)

Exercise 3.12. ( ) Prove Theorem 2.16: If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

Exercise 3.13. ( ) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Show that if f is continuous at x and g is continuous at $f(x)$, then the composition $g \circ f$ is continuous at x.

Exercise 3.14. ( ) Let f, g, h be functions $\mathbb{R} \to \mathbb{R}$ such that for all x, $f(x) \le g(x) \le h(x)$. Show that if $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = 0$, then $\lim_{x\to 0} g(x) = 0$. (This is known as the squeeze theorem.)

3.5 Some exercises with quantifiers

Exercise 3.15. ( ) Give examples to show that the following definitions of $\lim_{x\to c} f(x) = L$ are not correct.

1. $\forall \delta > 0, \exists \varepsilon > 0 \text{ s.t. } \forall x, \text{ if } 0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon$
2. $\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, \text{ if } |f(x) - L| < \varepsilon, \text{ then } 0 < |x - c| < \delta$

Exercise 3.16. ( ) Let $f(x) = x^2$. Are the following statements true?

1. $\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in \mathbb{R}, \forall c \in \mathbb{R}, \text{ if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \varepsilon$
2. $\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in [0,1], \forall c \in [0,1], \text{ if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \varepsilon$

Note that there are different from (2.8). To see how they differ, it might be helpful translating these statements into games.

(The relevant concept here is called *uniform continuity*. The two questions in this exercise can be rephrased as: (1) Is f uniformly continuous on $\mathbb{R}$? (2) Is f uniformly continuous on $[0, 1]$? We will not need this concept for the remainder of this week.)

4 Day 2

4.1 Limits are unique

Here is Theorem 2.16 again, now with a proof. See the scratch-work below Theorem 2.16 to motivate the proof given below.

Theorem 4.1. *If* $\lim_{x\to c} f(x) = L_1$ *and* $\lim_{x\to c} f(x) = L_2$, *then* $L_1 = L_2$

Proof. Suppose for contradiction that $L_1 \neq L_2$. Let $\varepsilon = \frac{1}{2}|L_1 - L_2|$.

Since $\lim_{x\to c} f(x) = L_1$, there exists a $\delta_1 > 0$ such that if $0 < |x - c| < \delta_1$, then $|f(x) - L_1| < \varepsilon$.

Since $\lim_{x\to c} f(x) = L_2$, there exists a $\delta_2 > 0$ such that if $0 < |x - c| < \delta_2$, then $|f(x) - L_2| < \varepsilon$.

Fix a single x_0 satisfying $0 < |x_0 - c| < \min(\delta_1, \delta_2)$. By the triangle inequality, $|L_1 - L_2| \leq |f(x_0) - L_1| + |f(x_0) - L_2| < 2\varepsilon = |L_1 - L_2|$. This is a contradiction. □

4.2 Basic limit properties

Theorem 4.2. $\lim_{x\to c}(ax + b) = ac + b$.

Proof. This was Exercise 3.3. □

Theorem 4.3. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$, *then* $\lim_{x\to c}[f(x) + g(x)] = L + M$.

Scratch-work. Our goal is to show $\lim_{x\to c}[f(x) + g(x)] = L + M$. Recall that this means:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, \text{ if } 0 < |x - c| < \delta \text{ then } |f(x) + g(x) - L - M| < \varepsilon \tag{4.1}$$

Let's play the game. Espeon gives us an ε.

What we observe is that by the triangle inequality, $|f(x) + g(x) - L - M| \leq |f(x) - L| + |g(x) - M|$. So maybe we should try to show that both $|f(x) - L|$ and $|g(x) - M|$ are less than $\varepsilon/2$.

We need to choose δ at this stage of our game. But let's put the $\lim_{x\to c}[f(x) + g(x)] = L + M$ game on hold for a second and play some other games first. However, let's remember the ε that was given to us.

Let's play the $\lim_{x\to c} f(x) = L$ game, where Epseon starts with $\varepsilon/2$ (instead of the usual ε). Since we know the limit is L, Delphox has a response to Espeon's $\varepsilon/2$, which we can call δ_1.

Similarly, if Espeon plays $\varepsilon/2$ in the $\lim_{x\to c} g(x) = M$ game, Delphox has some response, which we can call δ_2.

What is means is:

$$\text{if } 0 < |x - c| < \delta_1 \text{ then } |f(x) - L| < \frac{\varepsilon}{2} \tag{4.2}$$

$$\text{if } 0 < |x - c| < \delta_2 \text{ then } |g(x) - M| < \frac{\varepsilon}{2} \tag{4.3}$$

So back to our $\lim_{x\to c}[f(x) + g(x)] = L + M$ game. At this point we should choose $\delta = \min(\delta_1, \delta_2)$. (In terms of a picture, this is saying take the smaller of the two vertical strips.)

Proof. Let $\varepsilon > 0$.

Since $\lim_{x\to c} f(x) = L$, $\exists \delta_1 > 0$ s.t. $\forall x,$ if $0 < |x - c| < \delta_1$ then $|f(x) - L| < \varepsilon/2$.

Since $\lim_{x\to c} g(x) = M$, $\exists \delta_2 > 0$ s.t. $\forall x,$ if $0 < |x - c| < \delta_2$ then $|g(x) - M| < \varepsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$.

If if $0 < |x-c| < \delta$, then $|f(x)+g(x)-L-M| \le |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$ □

Theorem 4.4. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then* $\lim_{x\to c}[f(x)g(x)] = LM$*.*

Proof. This is an exercise. □

Theorem 4.5. *If* $\lim_{x\to c} f(x) = L$ *and* $L \neq 0$*, then* $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{L}$*.*

Proof. This is an exercise. □

From these properties combined, we can deduce that all polynomials are continuous. And all rational functions are continuous whenever their denominators are nonzero.

4.3 The case $c = \infty$ and the case $L = \infty$

How do we define $\lim_{x\to c} f(x) = L$ when $c = \infty$ or $L = \infty$?

It doesn't make sense to just use our definition above and substitute $c = \infty$ or $L = \infty$.

Think about the picture and what changes in the picture. Here's what we get:

Definition 4.6. $\lim_{x\to\infty} f(x) = L$ means: "$\forall\varepsilon > 0, \exists N$ s.t. $\forall x,$ if $x > N$ then $|f(x) - L| < \varepsilon$"

Definition 4.7. $\lim_{x\to c} f(x) = \infty$ means...? You should figure this out yourself! Exercise 5.1

(Here, the choices of the letter N is not standard. There are no standard choices, although I usually see one of K, M, N being used here. The capital letter suggests that these numbers are big)

I do not recommend memorizing these definitions. Instead, try to think about the picture, and translate that picture into a definition. Once you understand the epsilon-delta definition, it is easily modified into these cases involving infinity. You don't need to do any more work!

Observe that the theorem about limits of $f(x) + g(x)$, $f(x)g(x)$ and $1/f(x)$ all work in the case $x \to \infty$. The proofs are exactly the same.

Example 4.8. Let's show $\lim_{x\to\infty} \frac{1}{x} = 0$.

Scratch-work. Suppose $\varepsilon > 0$. We want $\frac{1}{x} < \varepsilon$. So we need $x > \frac{1}{\varepsilon}$. (Here's we're only looking at positive x. We don't care about what happens for negative x since we're taking $x \to \infty$.)

Proof. Suppose $\varepsilon > 0$. Then let $N = 1/\varepsilon$. Suppose $x > N$. Then $|\frac{1}{x} - 0| = \frac{1}{x} < \frac{1}{N} = \varepsilon$. □

4.4 Limits of sequences

A sequence of numbers is, well, a sequence of numbers. Usually a sequence is written like $a_1, a_2, a_3, \ldots$. Or we can abbreviate it in various ways, such as $(a_n)_{n=1}^\infty$ or $(a_n)_n$. (Of course, we don't have to start a sequence at $n = 1$.)

Another way to think about a sequence is that it is a function whose domain is the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. If you input n, the function outputs a_n.

Thinking about sequences as functions defined on the natural numbers is very helpful. To define $\lim_{n\to\infty} a_n = L$, we can use the same definition as before!

Definition 4.9. $\lim_{n\to\infty} a_n = L$ (or $a_n \to L$, for short) means:

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \in \mathbb{N}, \text{ if } n > N \text{ then } |a_n - L| < \varepsilon. \tag{4.4}$$

Also, we say that a sequence $(a_n)_n$ *converges* or *is convergent* if there exists an $L \in \mathbb{R}$ such that $a_n \to L$.

We changed some notation (x to n, and $f(x)$ to a_n), but this doesn't really change anything. We could have called our function $f(x)$, where $x \in \mathbb{N}$, but by convention sequences are often written a_n instead.

Another change we made was we wrote $\forall n \in \mathbb{N}$. This is to remind ourselves that the domain of the function is now $\mathbb{N}$ instead of $\mathbb{R}$. But we could have also written $\forall n$ by itself, with the implicit assumption that n is a natural number. The following is equivalent to the definition given above:

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n > N, |a_n - L| < \varepsilon. \tag{4.5}$$

4.5 Series

What is the definition of $\sum_{k=1}^\infty a_k$?

First, we can define the sequence of partial sums $s_n = \sum_{k=1}^n a_k$. Then we define $\sum_{k=1}^\infty a_k$ to be $\lim_{n\to\infty} s_n$.

In other words, infinite series are secretly limits of sequences! If we want to understand infinite series, we should first understand limits of sequences. We'll return to series later.

4.6 Sequences and series of functions

Reference: (Spivak, Chapter 24)

Sequences and series of numbers are nice, but we're more interested in sequences and series of functions. For example, Taylor series such as $e^x = \sum_{k=0}^\infty \frac{x^k}{k!}$.

Recall that an infinite series is just a limit of partial sums. So really, we can first look at limits of sequences of functions.

Example 4.10. Let f_n be a sequence of functions defined on $[0,1]$ defined by $f_n(x) = x^n$. Then $\lim_{n\to\infty} f_n(x) = g(x)$, where

$$g(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases} \tag{4.6}$$

In the above example, we say that f_n converges pointwise to g. Let's give a definition of this.

Definition 4.11. Let f_n be functions defined on some domain A. Then we say f_n *converges pointwise* to g if

$$\forall x \in A, \forall \varepsilon > 0, \exists N \text{ s.t. } \forall n, \text{ if } n > N, \text{ then } |f_n(x) - g(x)| < \varepsilon \tag{4.7}$$

Definition 4.12. Let f_n be functions defined on some domain A. Then we say f_n *converges uniformly* to g if

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n, \text{ if } n > N, \text{ then } \forall x \in A, |f_n(x) - g(x)| < \varepsilon \tag{4.8}$$

The pointwise convergence game:

1. Espeon starts by choosing $x \in A$ and $\varepsilon > 0$.
2. Delphox responds by choosing N.
3. Espeon responds by choosing $n > N$.
4. Delphox responds by showing that $|f_n(x) - g(x)| < \varepsilon$.

The uniform convergence game:

1. Espeon starts by choosing $\varepsilon > 0$.
2. Delphox responds by choosing N.
3. Espeon responds by choosing $n > N$ and $x \in A$.
4. Delphox responds by showing that $|f_n(x) - g(x)| < \varepsilon$.

The KEY difference is that in the pointwise convergence case, Delphox can see Espeon's choice of x before choosing N. However, in the uniform convergence case, Delphox must make a choice of N without knowing that Espeon's choice of x is. In fact, Espeon is allowed to choose x depending on Delphox's N to give Delphox a hard time.

5 Day 2 exercises

Exercise 5.1. ( ) Give a definition for $\lim_{x\to c} f(x) = \infty$.

Exercise 5.2. ( ) Show $\lim_{x\to\infty} \frac{1}{x^5+x+1} = 0$.

Exercise 5.3. ( ) Show $\lim_{x\to\infty} \sin x$ does not exist.

Exercise 5.4. () Define the sequence $a_n = (-1)^n$. Show $\lim_{n\to\infty} a_n$ does not exist.

Exercise 5.5. () Consider $a_n = 2^{-n}$.

1. Calculate the partial sums $s_n = \sum_{k=1}^{n} a_n$. (Hint: consider $2s_n$)
2. Give a rigorous proof that $\sum_{k=1}^{\infty} a_k = 1$.

Exercise 5.6. () We define a *Cauchy sequence* to be a sequence $(a_n)_n$ satisfying the following:

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall m, n > N, |a_n - a_m| < \varepsilon. \tag{5.1}$$

Prove that if a sequence $(a_n)_n$ of numbers converges, then it is a Cauchy sequence.

Exercise 5.7. () Prove Theorem 4.4: If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then $\lim_{x\to c}[f(x)g(x)] = LM$. (Hint: This is like Theorem 4.3, but you need to start with $f(x)g(x) - LM$ and somehow rewrite it so it contains the expressions $f(x) - L$ and $g(x) - M$. How do you do that?)

Exercise 5.8. () Prove Theorem 4.5: If $\lim_{x\to c} f(x) = L$ and $L \neq 0$, then $\lim_{x\to c} \frac{1}{f(x)} = \frac{1}{L}$.

Exercise 5.9. () Prove that $f(x)$ is continuous at $x = c$ if and only if for all sequences $(a_n)_{n=1}^{\infty}$, if $a_n \to c$, then $f(a_n) \to f(c)$. (Note: Ben says this fact will be useful in his classes.)

Exercise 5.10. () Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Exercise 5.11. () Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0$.

Exercise 5.12. () Suppose $\sum_{k=1}^{\infty} a_k$ converges. Let $N > 0$ and let $(b_n)_n$ be a sequence obtained by rearranging the first N terms of $(a_n)_n$ in some way, and leaving the rest untouched. (So for $n > N$, the two sequences are identifcal.) Prove that $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} b_k$.

5.1 Pointwise convergence is not good enough

You don't need to justify/prove any of the limits. I just want you to play around with some examples. It might help to think geometrically.

Exercise 5.13. () Find a sequence of functions $(f_n)_{n=1}^{\infty}$ defined on $[0, 1]$ with the following properties:

- Each f_n is Riemann integrable
- $(f_n)_n$ converges pointwise to g

- $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 g(x)\,dx$

Exercise 5.14. () Find a sequence of functions $(f_n)_{n=1}^{\infty}$ defined on $[0,1]$ with the following properties:

- Each f_n is continuous
- $(f_n)_n$ converges pointwise to g
- g is not continuous

Exercise 5.15. () Find a sequence of functions $(f_n)_{n=1}^{\infty}$ defined on $[0,1]$ with the following properties:

- Each f_n is differentiable
- $(f_n)_n$ converges pointwise to g
- $(f_n')_n$ does not converge pointwise to g'

5.2 Uniform convergence saves the day?

Exercise 5.16. () Suppose $(f_n)_n$ converges uniformly to g. Prove that $(f_n)_n$ converges pointwise to g.

Exercise 5.17. () Let $(f_n)_n$ and g be as in Example 4.10.

1. Does $(f_n)_n$ converge uniformly to g on $[0,1]$?
2. What about on $[0,\frac{1}{2}]$? (In other words, restrict the domains of f_n and g to $[0,\frac{1}{2}]$.)

Exercise 5.18. () If $(f_n)_n$ is a sequence of Riemann integrable functions defined on $[a,b]$ and f_n converges uniformly to g, then is it true that $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b g(x)\,dx$?

Exercise 5.19. () If $(f_n)_n$ is a sequence of continuous functions and f_n converges uniformly to g, then is it true g is continuous?

Exercise 5.20. () If $(f_n)_n$ is a sequence of differentiable functions and f_n converges uniformly to g, then is it true g is differentiable? Is it true that $(f_n')_n$ converges pointwise to g'? Converges uniformly?

6 Day 3

6.1 Pointwise vs uniform convergence

When you have a sequence of functions $(f_n)_n$, you can try to picture it as a movie. First you see the graph of $y = f_1(x)$. Then it disappears and you see the graph of $y = f_2(x)$. And so on. (So you can think of n as the "time" in this movie.)

Let's try to think about pointwise and uniform convergence of $(f_n)_n$ to g in terms of these "movies." (See also Section 4.6 for interpretations in terms of games.)

In pointwise convergence, you keep track of only one x-coordinate: someone (Espeon?) gives you two things: A point x_0 and an $\varepsilon > 0$. Draw the "wiggly strip" between $y = g(x) - \varepsilon$ and $y = g(x) + \varepsilon$. As the movie plays you keep track of only the point $(x_0, f_n(x_0))$ of the graph. Ignore the other x-coordinates. After some time, this point should be inside the wiggly strip.

In uniform convergence, you keep track of the entire function as it changes. some gives you $\varepsilon > 0$. Again, consider the "wiggly strip" between $y = g(x) - \varepsilon$ and $y = g(x) + \varepsilon$. After some time moment of your movie, the entire graph of $y = f_n(x)$ will always be inside this strip.

6.2 Uniform convergence and integration

Because of time, we're not going to rigorously define the Riemann integral of a function. It can be defined by limits, but we need a more general notion of limit than what we have so far, involving things called nets.

We will use the following property of integrals without proof. Triangle inequality for Riemann integral: $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

Theorem 6.1. *If $(f_n)_n$ is a sequence of Riemann integrable functions defined on $[a, b]$ and f_n converges uniformly to g, then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b g(x)\,dx. \tag{6.1}$$

In other words, we can interchange the limit and the integral:

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx \tag{6.2}$$

You can see the idea of the proof by drawing a picture.

Proof. Let $\varepsilon > 0$.

Since $(f_n)_n$ converges uniformly to g, there exists an N such that for all n, if $n > N$, then for all $x \in [a, b]$, $|f_n(x) - g(x)| < \frac{\varepsilon}{(b-a)}$.

For $n > N$, it follows that

$$\left|\int_a^b f_n(x)\,dx - \int_a^b g(x)\,dx\right| = \left|\int_a^b [f_n(x) - g(x)]\,dx\right| \tag{6.3}$$

$$\leq \int_a^b |f_n(x) - g(x)|\,dx \tag{6.4}$$

$$< \int_a^b \frac{\varepsilon}{(b-a)}\,dx \tag{6.5}$$

$$= \varepsilon \tag{6.6}$$

□

What allowed this proof to work is that we were able to pick a single N that works for all $x \in [a, b]$. If we only had pointwise convergence, we would not have been able to do line (6.5).

In fact, the theorem is false if we replace uniform convergence with pointwise convergence. (You were asked to show this in Exercise 5.14.) Here is a rough description of a counterexample:

Example 6.2. Consider a sequence of functions f_n on $[0, 1]$, such that each f_n has a large "spike" and that the spike moves towards 0 as $n \to \infty$. John Conway described this as a "tsunami." (This description probably doesn't make any sense unless you see the picture...) This sequence converges pointwise to the zero function, but the sequence $\int_0^1 f_n(x)\,dx$ does not converge to 0. If you fill in the details, this gives a solution to Exercise 5.14

Fun fact 6.3. There's actually a much more powerful version of Theorem 6.1 called the *dominated convergence theorem*: Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of functions such that

1. Each $(f_n)_n$ is Lebesgue integrable and the sequence converges pointwise to g.
2. There exists a nonnegative function h such that $\int_{-\infty}^{\infty} h(x)\,dx < \infty$ and such that for all n and x, $|f_n(x)| \leq h(x)$.

Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b g(x)\,dx \tag{6.7}$$

Note that you need to use the Lebsegue theory of integration (which is beyond the scope of this class). This theorem is false for Riemann integration.

6.3 Uniform convergence and continuity

Again, pointwise convergence is not enough to preserve continuity. Example 4.10 is an example where the functions f_n are continuous but the pointwise limit g is not. In fact, this example can also be used to create an example of a sequence of continuous functions h_n such that $\sum h_n(x)$ converges pointwise, but the function $\sum_{n=1}^{\infty} h_n(x)$ is not continuous. See Exercise 7.3.

Theorem 6.4 (Uniform limit theorem)**.** *If $(f_n)_n$ is a sequence of continuous functions defined on A and f_n converges uniformly to $g : A \to \mathbb{R}$, then g is continuous.*

Scratch-work. Here are some rough ideas.

So we want to be able to compare $g(x)$ with $g(c)$ and show that they are close. But all we know is that g is the uniform limit of the sequence (f_n). So here's the idea:

1. Show $g(c)$ is close to $f_n(c)$. We can use uniform convergence here.

2. Show $f_n(c)$ is close to $f_n(x)$. We can use the fact that f_n is continuous.

3. Show $f_n(x)$ is close to $g(x)$. We can again use uniform convergence. (In fact this step would be impossible if we only had pointwise convergence!)

I wasn't really precise with the meaning of "close"...

Proof. We'll prove this tomorrow, but you should try it on your own first. Exercise 7.7. □

6.4 Uniform convergence and differentiation

It turns out differentiation are much harder than integration and continuity. Uniform convergence of f_n was enough for both integration and continuity, but it turns out it is not enough for differentiation.

Here is a counterexample to show it is false.

Example 6.5. Consider a sequence of functions f_n which oscillate like a wave (e.g., like a sine or a cosine). As n increases, the amplitude of the wave decreases, but the frequency increases. If you do this correctly, then (f_n) converges uniformly to the zero function, but f_n' does not converge to the zero function, not even pointwise. You should fill in the details in Exercise 7.5.

Theorem 6.6. *Let $[a, b]$ be an interval. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions with the following properties.*

1. $(f_n)_n$ converges pointwise to some function f.

2. *For all n, f_n' exists and is continuous.*

3. *$(f_n')_n$ converges uniformly to some function g*

Then f is differentiable and $f' = g$.

(Note: There are ways to slightly weaken the conditions in the hypothesis and achieve the same conclusion. But in practice, the version I have stated is usually good enough. The third condition, about the uniform convergence of the sequence $(f_n')_n$ is essential.)

Proof. This is also an exercise. See Exercise 7.8. Because of time, we're not going to prove this one in class. See me if you'd like a hint or if you'd like me to explain the proof. □

7 Exercises

Exercise 7.1. ( ) Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n}x^2$.

1. Does f_n converge pointwise on $\mathbb{R}$?

2. Does f_n converge uniformly on $\mathbb{R}$?

3. Does f_n converge pointwise on $[0, 100]$?

4. Does f_n converge uniformly on $[0, 100]$?

Exercise 7.2. ( ) Repeat Exercise 7.1 but with $f_n(x) = \frac{1}{n}\sin x$.

7.1 Some important counterexamples

The counterexamples in this section answer some of the questions posed in Section 2.1.1.

Exercise 7.3. ( ) Use Example 4.10 to construct an example of a sequence of continuous functions $h_n : [0, 1] \to \mathbb{R}$ such that $\sum h_n(x)$ converges pointwise to a function that is not continuous.

Exercise 7.4. ( ) First, do Exercise 5.14 if you haven't already. (Hint: consider a function as described in Example 6.2.)

Now use Exercise 5.14 to construct an example of a sequence of Riemann integrable functions $h_n : [0, 1] \to \mathbb{R}$ such that $\sum h_n(x)$ converges pointwise but

$$\int_0^1 \sum_{n=1}^{\infty} h_n(x)\,dx \neq \sum_{n=1}^{\infty} \int_0^1 h_n(x)\,dx \tag{7.1}$$

You should feel free to any properties of the integral that you know. We don't have time to develop the theory of Riemann integration from scratch.

Exercise 7.5. ( ) Fill in the details of Example 6.5. Use the description there to write down a sequence of differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ such that f_n converges uniformly to the zero function, but f_n' does not. (Note that this also solves Exercise 5.20.)

Exercise 7.6. ( ) Use Exercise 7.5 to construct an example of a sequence of Riemann integrable functions $h_n : [0, 1] \to \mathbb{R}$ such that $\sum h_n(x)$ converges pointwise but

$$\frac{d}{dx} \sum_{n=1}^{\infty} h_n(x) \neq \sum_{n=1}^{\infty} h_n'(x) \tag{7.2}$$

You should feel free to any properties of the derivative that you know.

7.2 Uniform convergence, continuity, and differentiation

Exercise 7.7. ( ) Prove Theorem 6.4. It might help to read the "scratch-work" section right after the statement of the theorem.

Exercise 7.8. ( ) Prove Theorem 6.6. Hint: The full proof is actually very short, but that doesn't mean it's easy to find.

7.3 Random problem

Exercise 7.9. ( ) Can you find an example of a single function $f : \mathbb{R} \to \mathbb{R}$ with the following properties?

1. f is differentiable.
2. $f'(0) = 0$.
3. $\lim_{x \to 0} f'(x)$ does not exist.

(I'm including this problem because it can be solved by adapting some ideas of one of the counterexamples we saw.)

8 Day 4

8.1 Uniform convergence and continuity, the proof

Now we prove the uniform limit theorem (Theorem 6.4). Read the scratch-work following the statement Theorem 6.4 for a motivation of the proof presented here.

Proof. Let $\varepsilon > 0$ and let $c \in A$.

We will show g is continuous at c.

By uniform convergence there is a N such that for all $n > N$ and all $x \in A$, $|f_n(x) - g(x)| < \varepsilon/3$.

Fix a single $n > N$. Since f_n is continuous at c, there is a $\delta > 0$ such that for all x, if $|x - c| < \delta$, then $|f_n(x) - f_n(c)| < \varepsilon/3$.

Suppose $|x - c| < \delta$. By the triangle inequality,

$$\begin{aligned} |g(x) - g(c)| &= |g(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - g(c)| && (8.1) \\ &\leq |g(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - g(c)| && (8.2) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} && (8.3) \\ &= \varepsilon && (8.4) \end{aligned}$$

□

Protip 8.1. This is an example of what is sometimes called a "$\varepsilon/3$-argument." The idea is this: Suppose you want to show a and b are close. But you can't do this directly. So you show a is close to c_1, which is close to c_2, which is close to..., which is close to c_n, which is close to b. Since being close is kind of transitive (not quite – the errors add up), it follows that a is close to b.

In the example above, you're comparing $n+1$ pairs of points. So this would result in an "$\varepsilon/(n+1)$-argument."

Fun fact 8.2. First let's define uniform continuity. (See also Exercise 3.16).

A function $f : A \to \mathbb{R}$ is *uniformly continuous* if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, \forall c \in A, \text{ if } |x - c| < \delta \text{ then } |f(x) - f(c)| < \varepsilon \qquad (8.5)$$

The difference between the definitions of continuity and uniform continuity is that the "$\forall c \in A$" gets moved to a later part of the statement (kind of like what happened with pointwise convergence and uniform convergence.)

Do not confuse *uniform continuity* with *uniform convergence*. Uniform continuity is a property of a single function. Uniform convergence is a property of a sequence of functions (which don't even have to be continuous).

The proof of Theorem 6.4 above can be easily adapted to show the following: If (f_n) is a sequence of uniformly continuous functions that converges uniformly to g, then g is uniformly continuous. See Exercise 9.15.

8.2 Completeness of the real numbers

Often we care more about whether or not a sequence/series converges, and we care less about what it actually converges to.

Definition 8.3. We say a_n converges, or $\lim_{n\to\infty} a_n$ exists, if $\exists L$ s.t. $a_n \to L$.

But wait, how do you show something converges without knowing what it converges to?

For example, for a long time, mathematicians knew that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converged, but they didn't know what it converged to.

Fun fact 8.4. By the integral test (or the Cauchy condensation test, see Exercise 9.7), we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The Basel problem asked if there is a nice expression for this series. Euler solved the problem in 1734 by showing that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. (One way to prove this is to use Fourier series.)

The similar looking expression $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges as well. The value is called Apéry's constant. Over 200 years after Euler solved the Basel problem, Apéry proved that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is irrational. We currently don't know if there's a nice expression for this constant.

This brings us to an important property of the real numbers, called the monotone convergence theorem.

Theorem 8.5 (Monotone convergence theorem for real numbers)**.** *Suppose (a_n) is a sequence of real numbers that is*

1. *monotone ($a_1 \le a_2 \le a_3 \le \cdots$ or $a_1 \ge a_2 \ge a_3 \ge \cdots$), and*
2. *bounded ($\exists M$ s.t. $\forall n, |a_n| \le M$)*

Then a_n converges to some real number L.

Proof. Actually, let's take this as an axiom. See the fun fact below for more. □

Fun fact 8.6. If we had lived in some world where only rational numbers existed, then everything we did would have worked fine. But this monotone convergence theorem is a way to distinguish the rationals from the reals. The monotone convergence theorem is not true for rationals! A bounded and monotone sequence of rational numbers does not necessarily converge to some real number! For example if we take decimal approximations to $\sqrt{2}$ (rounded down), we get the sequence $1, 1.4, 1.41, 1.414, \ldots$. This is a sequence of rational numbers that is bounded and monotone but does not converge to a rational number!

A way to think about this is that the rationals are full of "holes." which are filled in when you add in the real numbers.

Another way to talk about completeness of the reals is via Cauchy sequences. In fact, this idea generalizes to other metric spaces (whereas the monotone convergence theorem is harder to generalize). I wish I had time to talk about Cauchy sequences in the class. Unfortunately, I don't, so you'll just have to refer to Exercise 5.6, Exercise 9.12, Exercise 9.13.

One very useful consequence of the monotone convergence theorem is that any series which only has nonnegative terms and whose partial sums are bounded is convergent.

Theorem 8.7 (A test for series with nonnegative terms)**.** *Let a_n be a sequence of nonnegative numbers. Let $s_n = \sum_{k=1}^{n} a_k$ be the sequence of its partial sums. Then $\sum a_n$ converges if and only if the sequence s_n is bounded (i.e., there exists M such that for all n, $s_n \leq M$).*

Proof. Since $a_n \geq 0$, the sequence of partial sums is nondecreasing ($s_1 \leq s_2 \leq s_3 \leq \cdots$). The result follows from the monotone convergence theorem applied to the sequence $(s_n)_n$. □

8.3 The only two series convergence tests you need

When John Conway taught my analysis class, he claimed that you only needed two convergence tests: the comparison test and Dirichlet's test. From my experience this is indeed true. Most of the other tests for series convergence follow easily from one of these two. Well, actually Theorem 8.7 helps too.

First up is the comparison test (sometimes called the "direct comparison test").

Theorem 8.8 (Comparison test)**.** *Suppose (a_n) and (b_n) are sequences with $|a_n| \leq b_n$, and suppose that $\sum b_n$ converges. Then $\sum a_n$ converges.*

Proof. See Exercise 9.2. □

Next up is Dirichlet's test. (John Conway referred to this as the "damped oscillation test." You can think of the $(b_k)_k$ as "oscillating" and the a_k as "dampening.") This one is probably less familiar.

Theorem 8.9 (Dirichlet's test)**.** *Let (a_n) and (b_n) be sequences of numbers satisfying:*

1. *a_n is a positive and nonincreasing sequence, i.e. $a_1 \geq a_2 \geq \cdots$.*
2. *$\lim_{n\to\infty} a_n = 0$.*
3. *There is a constant M such that for all n, $|\sum_{k=1}^{n} b_k| \leq M$.*

Then $\sum a_n b_n$ converges.

Proof. See Exercise 9.4. □

We won't need Dirichlet's test in this class, but I'll point out that it contains the alternating series test as a special case. (See Exercise 9.5.)

8.4 Weierstrass M-test

The Weierstrass M-test is like the comparison test, but for series of functions. It is a way to test if a series of functions is uniformly convergent.

Theorem 8.10 (Weierstrass M-test)**.** *Let $(f_n)_n$ be a sequence of functions. Suppose that there exists a sequence of numbers $(M_n)_n$ such that*

1. *For all n and all x, $|f_n(x)| \leq M_n$*
2. *$\sum M_n$ converges*

Then the series $\sum f_n(x)$ converges uniformly.

Proof. We'll prove this tomorrow, but you can try this as an exercise. Exercise 9.8. □

9 Day 4 exercises

9.1 Convergence of sequences and series of numbers

Exercise 9.1. ( ) Give a counterexample to show that the following statement is false.

"Let a_n be a sequence of numbers. Let $s_n = \sum_{k=1}^{n} a_k$ be the sequence of its partial sums. Suppose that there exists M such that for all n, $|s_n| \leq M$. Then $\sum a_n$ converges."

This shows that the assumption of nonnegativity is very important in Theorem 8.7.

Exercise 9.2. ( ) Prove the comparison test (Theorem 8.8). Hint: The key observation is that $a_n + b_n \geq 0$.

Exercise 9.3. ( ) We say that $\sum a_n$ is *absolutely convergent* if $\sum_n |a_n|$ converges. Suppose that $\sum a_n$ is absolutely convergent. Prove the following.)

1. $\sum_n a_n$ converges (This shows that absolute convergence implies convergence.)
2. $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$. (This is known as the triangle inequality for infinite series.)

Exercise 9.4. ( ) This exercise will outline the proof of Dirichlet's test of convergence (Theorem 8.9):

1. Let $B_n = \sum_{k=1}^{n} b_k$. Show that

$$\sum_{k=1}^{n} a_k b_k = a_n B_n + \sum_{k=1}^{n-1} (a_k - a_{k+1}) B_k. \tag{9.1}$$

This is called the *summation by parts* formula. It is a discrete analogue of the integration by parts formula. (B_n is the "integral" of b_n, and $a_{n+1} - a_n$ is the "derivative" of a_n.)

2. Use summation by parts to prove Dirichlet's test.

Exercise 9.5. () Using Dirichlet's test (Theorem 8.9), prove the alternating series test: If c_n is a positive, nondecreasing series, then $\sum(-1)^n c_n$ converges if and only if $\lim_{n\to\infty} c_n = 0$.

Exercise 9.6. () Prove the other comparison tests you learned in calculus (e.g. limit comparison test, root test, ratio test, integral test). You should be able to do all of these with the comparison test, Dirichlet's test, or the test for series with nonnegative terms (Theorem 8.7). (Hint: For many of these you probably should compare with geometric series.)

Exercise 9.7. () The Cauchy condensation test is perhaps less well-known to calculus students. It can often be used in place of the integral test. Here is the statement: If $(a_n)_n$ is a nonnegative and non-increasing sequence of of numbers ($a_0 \geq a_1 \geq a_2 \geq \cdots \geq 0$), then $\sum a_n$ converges if and only if $\sum 2^n a_{2^n}$ converges. Prove it.

(Hint: How can you prove the harmonic series $\sum \frac{1}{n}$ diverges without using the integral test?)

9.2 Convergence of sequences and series of functions

Exercise 9.8. () Prove the Weierstrass M-test (Theorem 8.10).

Exercise 9.9. () Prove that the series $\sum_{k=0}^{\infty} x^k$ converges uniformly on the closed interval $[-\frac{1}{2}, \frac{1}{2}]$. (Hint: Weierstrass M-test.)

Exercise 9.10. () Show the series $\sum_{k=0}^{\infty} x^k$ does not converge uniformly on the open interval $(-1, 1)$. (Hint: compute the partial sums)

Exercise 9.11. () For $\alpha > 0$, define the function $f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha}\cos(2^n x)$. Prove that

1. If $\alpha > 0$, then f_α is continuous.

2. If $\alpha > 1$, then f_α is differentiable.

Hint: Use the Weierstrass M-test together with the results about uniform convergence (Theorem 6.4 and Theorem 6.6).

Fun fact: When $0 < \alpha \leq 1$, something weird happens. Look up "Weierstrass functions" on Wikipedia for some pictures.

Another note: The functions $2^{-n\alpha}\cos(2^n x)$ that we're summing up are waves. As n increases, the amplitude decreases and the frequency increases, so this is very similar to the situation described in Example 6.5. Except there, we didn't sum up the functions.

9.3 Completeness of the reals

This is kind of a side-topic. It's not a focus of this class.

Exercise 9.12. () Recall the definition of a Cauchy sequence given in (5.1).

Prove that a sequence $(a_n)_n$ of real numbers is convergent if and only if it is a Cauchy sequence. (Note: One direction is Exercise 5.6. You only need to prove the other direction here.)

Hint: Use the monotone convergence theorem (Theorem 8.5). The sequence $(a_n)_n$ may not be monotone though. Can you find a way to construct a new sequence from $(a_n)_n$ which is monotone?

Remark: You can also start with the statement that "convergent iff Cauchy" and use it to deduce the monotone convergence theorem. So this statement works equally well as an axiom of the real numbers, and some textbooks do things this way.

Exercise 9.13. () Show that Exercise 9.12 is false if we replace $\mathbb{R}$ with $\mathbb{Q}$.

Exercise 9.14. () Recall that $\mathbb{Q}$ denotes the set of rational numbers. Define the function $f : \mathbb{Q} \to \mathbb{R}$ as follows.

$$f(x) = \begin{cases} 0 & \text{if } x \leq \sqrt{2} \\ 1 & \text{otherwise} \end{cases} \qquad = \begin{cases} 0 & \text{if } x \leq 0 \text{ or } x^2 \leq 2 \\ 1 & \text{otherwise} \end{cases} \tag{9.2}$$

Recall the definition of continuity in Definition 2.19. Prove that f is continuous. This shows that the intermediate value theorem fails for $\mathbb{Q}$.

9.4 Uniform continuity

We don't need uniform continuity in this class, but this might be helpful for other analysis classes. (See Fun fact 8.2 for a short discussion of uniform continuity.)

Exercise 9.15. () Prove that if $f_n : A \to \mathbb{R}$ is a sequence of uniformly continuous functions that converges uniformly to g, then g is uniformly continuous. Hint: Adapt the proof of Theorem 6.4. If you understand the proof as well as the definitions of uniform continuity and uniform convergence, it should not be too hard.

Exercise 9.16. () Let $[a, b]$ be a (closed and bounded) interval. Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous. This is a special case of something called the Heine–Cantor theorem.

Hint: This is one of those statements that is true for reals but not rationals. (Can you find a counterexample for the rationals?) So you will need to use completeness of the reals somewhere

Another hint: There are several ways to approach this problem. Here is one possible way:

1. First, a definition: Let $\mathcal{S}$ be a set of open intervals of $\mathbb{R}$. (Each element of $\mathcal{S}$ is an open interval (c, d). Note that $\mathcal{S}$ could be a finite set, a countable set, or an uncountable set. There are no restrictions.) We say that "$\mathcal{S}$ covers $[a, b]$" if $[a, b] \subset \bigcup_{(c,d)\in\mathcal{S}}(c, d)$.

2. OK, this is the real first step: Prove that if $\mathcal{S}$ is a set of open intervals of $\mathbb{R}$ that covers $[a, b]$, then there is a finite subset $\mathcal{S}' \subset \mathcal{S}$ such that $\mathcal{S}'$ still covers $[a, b]$. (Or you could take this for granted and continue to the next step.)

3. Use the previous step to prove the desired result.

10 Day 5

10.1 Proof of Weierstrass M-test

First let's give a solution to Exercise 5.11.

Theorem 10.1. *If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = 0$.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. Let $L = \lim_{n\to\infty} s_n$. Then $L - s_n = \sum_{k=n+1}^{\infty} a_k$ (this actually requires proof). So

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} a_k = \lim_{n\to\infty} (L - s_{n-1}) = L - \lim_{n\to\infty} s_{n-1} = L - L = 0. \tag{10.1}$$

□

Now we prove the Weierstrass M-test (Theorem 8.10).

Proof of the Weierstrass M-test. First of all, if we fix x_0 and apply the comparison test to the sequence of numbers $(f_n(x_0))_n$, then we see that $\sum f_n(x_0)$ converges. Hence, $\sum f_n(x)$ converges pointwise to some function $g(x)$.

Now we need to show this convergence is uniform.

Let $\varepsilon > 0$.

Since $\sum M_n$ converges, by Theorem 10.1, there exists an N such that if $n > N$, then $\sum_{k=n}^{\infty} M_k < \varepsilon$.

Then if $n > N$, we have

$$\left|g(x) - \sum_{k=1}^{n} f_k(x)\right| = \left|\sum_{k=n+1}^{\infty} f_k(x)\right| \tag{10.2}$$

$$\leq \sum_{k=n+1}^{\infty} |f_k(x)| \tag{10.3}$$

$$\leq \sum_{k=n+1}^{\infty} M_k \tag{10.4}$$

$$< \varepsilon \tag{10.5}$$

□

10.2 Applications of the Weierstrass M-test

10.2.1 Power series

The Weierstrass M-test is extremely useful for power series.

Example 10.2. Consider the series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. (Let's pretend we don't already know that this is the function e^x.)

Let's restrict to only $x \in [-\frac{1}{2}, \frac{1}{2}]$. Then we have $|\frac{x^n}{n!}| \leq |x^n| \leq \frac{1}{2^n}$. So if we let $M_n = \frac{1}{2^n}$, then by the Weierstrass M-test (and the fact that $\sum M_n$ converges), it follows that the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on $[-\frac{1}{2}, \frac{1}{2}]$. In other words, the sequence of partial sums $s_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$ converges uniformly to $f(x)$.

Since each individual term $\frac{x^n}{n!}$ is continuous, the partial sums $s_n(x)$ are continuous, so by the uniform limit theorem (Theorem 6.4), f is continuous on $[-\frac{1}{2}, \frac{1}{2}]$.

Furthermore, $s_n'(x) = s_{n-1}(x)$, so by what we already showed above, the sequence $s_n'(x)$ also converges uniformly to f. Therefore, Theorem 6.6, f is differentiable on $[-\frac{1}{2}, \frac{1}{2}]$ and $f' = f$ on that interval.

What happens outside of $[-\frac{1}{2}, \frac{1}{2}]$? Actually, the series converges uniformly on any bounded subset of $\mathbb{R}$. You are asked to show this in Exercise 11.2.

(Ignore the following theorem if you're not familiar with power series.)

Theorem 10.3. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Let R be the radius of convergence of the power series. Let $r \in (0, R)$. Then the two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ both converge uniformly on $[-r, r]$.*

Proof. I'm not sure if I should assign this as an exercise. It requires some things we haven't covered in the class (such as the definition of radius of convergence). But I'll point out that this proof uses the Weierstrass M-test twice, once for each series. □

Corollary 10.4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Let R be the radius of convergence of the power series. Then f is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$*

Proof. Deducing this from Theorem 10.3 is an exercise. □

10.2.2 Continuous but nowhere differentiable functions

The Weierstrass M-test can also be used to construct some very bad functions.

Theorem 10.5. *For $\alpha > 0$, define the function $f_\alpha(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos(2^n x)$.*

1. *If $\alpha > 0$, then f_α is continuous.*
2. *If $\alpha > 1$, then f_α is differentiable.*

Proof. This was Exercise 9.11. □

Fun fact 10.6. The theorem does not say anything about the differentiability of f when $0 < \alpha \leq 1$. It turns out that if $0 < \alpha \leq 1$, then f_α is actually *not differentiable at any point of* $\mathbb{R}$. Unfortunately, this is beyond the scope of this course. The proof that I know uses some Fourier analysis. In any case, for $0 < \alpha \leq 1$, the functions f_α are functions that are continuous but nowhere differentiable! These are known as Weierstrass functions. (Look them up on Wikipedia to see some pictures.)

In general, *constructing* continuous but nowhere differentiable functions is hard. Exercise 11.3 gives an example that is easier than the Weierstrass functions above, but it's still hard.

It turns out that if you want to just prove the *existence* of continuous but nowhere differentiable functions, it's much easier! You can use something called the Baire category theorem. (See Ben's Week 3 class.)

10.3 Rearranging terms in a series

Definition 10.7. We say that $\sum a_n$ is *absolutely convergent* if $\sum_n |a_n|$ converges.

By Exercise 9.3, if a series is absolutely convergent, then it is convergent. But the converse is not true. For example, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges (by the alternating series test), but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. We say that $\sum \frac{(-1)^{n+1}}{n}$ is "conditionally convergent." (definition below)

Definition 10.8. We say that $\sum a_n$ is *absolutely convergent* if $\sum_n a_n$ converges but $\sum_n |a_n|$ does not.

10.3.1 Conditionally convergent series

Now we can get to a theorem, which tells us that conditionally convergent series are very crazy and we have to be careful with them.

Theorem 10.9 (Riemann's rearrangement theorem)**.** *Let $\sum a_n$ be a conditionally convergent series. Then for all $L \in \mathbb{R}$, it is possible to rearrange the terms into a new series which converges to L.*

Proof. (This proof makes much more sense with a picture.)

Let's give a proof for just the alternating harmonic series: $\sum \frac{(-1)^{n+1}}{n}$. If you inspect the proof, you see that it can be generalized to all conditionally convergent series.

From $a_n = \frac{(-1)^{n+1}}{n}$, we're going to split this into two sequences.

1. The positive terms: Let $p_n = \frac{1}{2n-1}$
2. The negative terms: Let $q_n = -\frac{1}{2n}$.

Note that both $p_n \to 0$ and $q_n \to 0$, but both $\sum p_n$ and $\sum q_n$ diverge.

Fix $L \in \mathbb{R}$. Now I'm going to describe an algorithm to rearrange a_n into a new sequence b_n such that $\sum_{n=1}^{\infty} = L$. For simplicity, let's consider just $L > 0$. (We can modify this argument so it works for $L \leq 0$.

The idea is to add terms to the sequence $(b_n)_n$ one by one, and keep track of the partial sums. At the start we don't have any terms in the sequence.

- **Step 1**: Take terms from the positive sequence $(p_n)_n$ and keep adding them into the sequence $(b_n)_n$ until the sum of the all numbers added to $(b_n)_n$ is greater than L. This is possible because $\sum_{n=1}^{\infty} p_n = \infty$. Stop doing this as soon as the sum is greater than L. So right now, the sequence (b_n) that we're building looks like:

$$p_1, p_2, \ldots, p_{N_1} \tag{10.6}$$

  where

$$\sum_{k=1}^{N_1-1} p_k \leq L \tag{10.7}$$

$$\sum_{k=1}^{N_1} p_k > L. \tag{10.8}$$

- **Step 2**: Take terms from the negative sequence $(q_n)_n$ and keep adding them into the sequence $(b_n)_n$ until the sum of the all numbers in $(b_n)_n$ is less than L. This is possible because $\sum_{n=1}^{\infty} q_n = -\infty$. Stop doing this as soon as the sum is less than L. Right now, the sequence (b_n) we're building looks like

$$p_1, p_2, \ldots, p_{N_1}, q_1, q_2, \ldots, q_{M_1} \tag{10.9}$$

where

$$\sum_{k=1}^{N_1} p_k + \sum_{k=1}^{M_1-1} q_k \geq L \tag{10.10}$$

$$\sum_{k=1}^{N_1} p_k + \sum_{k=1}^{M_1} q_k < L. \tag{10.11}$$

- **Step 3**: Go back to the positive sequence p_n. Starting at $n = N_1 + 1$ (where we left off), take terms from the positive sequence $(p_n)_n$, and keep adding them into the sequence $(b_n)_n$ until the sum of the all numbers added to $(b_n)_n$ is greater than L. This is possible because $\sum_{n=N_1+1}^{\infty} p_n = \infty$. Stop doing this as soon as the sum is greater than L. So right now, the sequence (b_n) that we're building looks like:

$$p_1, p_2, \ldots, p_{N_1}, q_1, q_2, \ldots, q_{M_1}, p_{N_1+1}, p_{N_1+2}, \ldots, p_{N_2} \tag{10.12}$$

where

$$\sum_{k=1}^{N_1} p_k + \sum_{k=1}^{M_1} q_k + \sum_{k=N_1+1}^{N_2-1} p_k \leq L \tag{10.13}$$

$$\sum_{k=1}^{N_1} p_k + \sum_{k=1}^{M_1} q_k + \sum_{k=N_1+1}^{N_2} p_k > L \tag{10.14}$$

- **Step 4, 5, 6, . . .**: Continue in the same way

Now we need to show that $\sum_{k=1}^{\infty} b_k = L$. (For the discussion that follows, it helps to draw a picture.)

First, consider what happens after Step 1. Our rearranged sequence so far is $p_1, p_2, \ldots, p_{N_1}$. Let $S = p_1 + \cdots + p_{N_1}$ be the total sum so far. We know that P is greater than L (thanks to (10.8)), but not by much (thanks to (10.7)). in particular $S \leq L + p_{N_1}$.

Now a crucial observation: *After this point in the sequence, the sum is never greater than* $L + p_{N_1}$. This is because the sequence $(p_n)_n$ is a nonnegative sequence that decreases to zero.

Next, consider what happens after Step 3. (We're skipping Step 2 for now). Our sequence is now (10.12). Let S be the total sum. By the same reasoning as before $S \leq L + p_{N_2}$. Furthermore, after this point in the sequence, the sum is never greater than $L + p_{N_2}$.

The same reasoning works for Steps 5, 7, 9, ...

Now let's move to the even steps, where we add in negative numbers to our sequence. Consider what happens after Step 2, where our sequence is now (10.12). If S denotes the total sum, then $S \geq L - |q_{M_1}|$.

The same reasoning works for Steps 4, 6, 8, ...

Since (p_n) and $(|q_n|)$ are both positive sequences that decrease to zero, it follows that the partial sums approach L (by the squeeze theorem). □

10.3.2 Absolutely convergent series

The previous section showed why conditionally convergent series are scary. They're like wild animals. This section shows absolutely convergent series are much better.

Here is one way that absolutely convergent series are nicer. The following is true for all convergent series $\sum a_n$ (not just absolutely convergent ones):

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n > N, \left| \sum_{k=n}^{\infty} a_k \right| < \varepsilon \tag{10.15}$$

(The statement above is Exercise 5.11.) However, the following is only true for absolutely convergent series:

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall \text{ subsets } S \subset \{n : n > N\}, \left| \sum_{k \in S}^{\infty} a_k \right| < \varepsilon \tag{10.16}$$

We can intuitively think of (10.15) as saying that for a convergent series, the "tails" are small. On the other hand (10.16) says that for absolutely convergent series, not only are the tails small, but any subset of the tails are small as well. This is false for conditionally convergent series! (Think about $\sum \frac{(-1)^{n+1}}{n}$ to see why.)

Theorem 10.10. *If (a_n) converges absolutely, and (b_n) is any rearrangement of (a_n), then $\sum b_n$ converges absolutely, and*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n. \tag{10.17}$$

Proof. Let $s_n = \sum_{k=1}^{n} a_n$ and $t_n = \sum_{k=1}^{n} b_n$. Let $S = \sum_{k=1}^{n} a_n$. We're going to show $t_n \to S$. That is, we'll show $\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n > N, |t_n - S| < \varepsilon$.

Suppose $\varepsilon > 0$. (Keep in mind that our goal is to now find an N.)

Since $\sum a_n$ converges absolutely, there is a $K > 0$ such that $\sum_{n=K+1}^{\infty} |a_n| < \varepsilon/2$. Note that this implies $|s_K - S| < \varepsilon/2$ as well.

Choose N so large that the finite sequence $b_1, b_2, \dots, b_N$ contains all the terms $a_1, a_2, \dots, a_K$.

We claim that this N works, i.e., in what follows we'll show $\forall n > N, |t_n - S| < \varepsilon$.

Suppose $n > N$. Then as mentioned above, all the terms in the partial sum s_K are contained in the terms of the partial sum t_n. Thus $t_n - s_K$ is a finite sum of some of the a_i, where we know for sure that $a_1, \ldots, a_K$ are not present. (Here we are rearranging terms in $t_n - s_K$, which is fine, since t_n and s_K are both finite sums.) Thus, by the triangle inequality,

$$|t_n - s_K| \leq \sum_{k=K+1}^{\infty} |a_k| < \frac{\varepsilon}{2} \tag{10.18}$$

By the triangle inequality again,

$$|t_n - S| \leq |t_n - s_K| + |s_K - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{10.19}$$

This completes the proof. □

11 Day 5 exercises

The class is over. But feel free to ask me questions about any of the following or previous exercises at any time during Mathcamp. And feel free to ask any questions about this class or about math in general! A good book for all this material, and for learning more, is Spivak's *Calculus* (not to be confused with his *Calculus on manifolds*).

11.1 Power series

Exercise 11.1. ( ) Prove Corollary 10.4. Be a little careful. There are some subtle technical details.

Exercise 11.2. ( ) Define $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. (Yes, this is the exponential function. But don't use that fact in this problem. That would be cheating!)

- Show that f is continuous on $\mathbb{R}$. (Hint: Don't consider f on all of $\mathbb{R}$ at once. Instead, temporarily fix some $R > 0$ and only consider the interval $[-R, R]$.)
- Show that f is differentiable on $\mathbb{R}$ and that $f' = f$.

11.2 Continuous but nowhere differentiable functions

Exercise 11.3. ( ) For $x \in \mathbb{R}$, let $g(x)$ the distance from x to the nearest integer. Prove that the function

$$f(x) = \sum_{n=1}^{\infty} \frac{g(10^n x)}{10^n} \tag{11.1}$$

defines a function that is continuous everywhere but differentiable nowhere.

11.3 Rearranging series

Exercise 11.4. () Let $\sum a_n$ be a conditionally convergent series. Show that you can rearrange the terms into a new series such that $\sum_{n=1}^{\infty} b_n = \infty$.

12 [Note to self] Changes for future iterations

1. Exercise for Day 1: Let $a, b \in \mathbb{R}$. Suppose that $\forall \varepsilon > 0, |a - b| < \varepsilon$. Prove that $a = b$.
2. Exercise for Day 1: Suppose $f(x) \leq g(x)$ and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Prove that $L \leq M$.