MATH 450, HOMEWORK 2

DUE FEBRUARY 12, 2016

Part I. Theory

Problem 1. Consider the theta method,

$$y_{n+1} = y_n + h\big[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\big].$$

- **a.** Find the domain of linear stability $\mathcal{D}_\theta \subset \mathbb{C}$. Sketch this in the complex plane for $\theta = 1/3$ and $\theta = 2/3$.
- **b.** For which values $\theta \in [0,1]$ is the method A-stable? Give a proof.

Problem 2 (Iserles, Exercise 2.4)**.** Determine the order of the three-step method

$$y_{n+3} - y_n = h\big[\tfrac{3}{8} f(t_{n+3}, y_{n+3}) + \tfrac{9}{8} f(t_{n+2}, y_{n+2}) + \tfrac{9}{8} f(t_{n+1}, y_{n+1}) + \tfrac{3}{8} f(t_n, y_n)\big],$$

the *three-eighths* scheme. Is it convergent?

Part II. Programming

Download the sample code `hw2.py`, which contains implementations of the Euler and backward Euler methods for *systems* of ODEs. (Last week's code was only designed for scalar ODEs.) The backward Euler code uses the nonlinear root-finder `fsolve` from the `scipy.optimize` library to solve for y_{n+1} at each step. Specifically,

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) \quad \Leftrightarrow \quad 0 = -y_{n+1} + y_n + hf(t_{n+1}, y_{n+1}),$$

so at each step, we can use `fsolve` to find a root of the function

$$F(y_{n+1}) = -y_{n+1} + y_n + hf(t_{n+1}, y_{n+1}),$$

with initial guess y_n.

Problem 3. Consider the simple harmonic oscillator

$$x'' = -x,$$

which can be written as the first-order linear system

$$\begin{pmatrix} x' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}.$$

- **a.** Apply the Euler method to this problem for $t \in [0, 100]$, with initial condition $\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Plot $x(t)$ vs. t for $h = 0.1$ and $h = 0.01$.
- **b.** Repeat part (a) for the backward Euler method.

Problem 4. Create a function `trapezoid(f, t0, y0, h, N)` that implements the trapezoid method. (Hint: This function should be very similar to `backwardEuler`, but with a different choice of `F`.) Use this to repeat Problem 3(a) for the trapezoid method.