Beurling primes and Hardy spaces of Dirichlet series

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Given an arbitrary increasing sequence $q = \{q_n\}_{n\geq 1}$, $1 < q_n \to \infty$, such that $\{\log q_n\}_{n\geq 1}$ is linearly independent over $\mathbb{Q}$, we will denote by $\mathbb{N}_q = \{\nu_n\}_{n\geq 1}$ the set of numbers that can be written (uniquely) as finite products with factors from q, ordered in an increasing manner. The numbers q_n are known as Beurling primes, and the numbers ν_n are Beurling integers. The corresponding generalized Dirichlet series are of the form

$$f(s) = \sum_{n\geq 1} a_n \nu_n^{-s}.$$

Bohr's theorem

(Joint work with Karl–Mikael Perfekt)

In the classical case, where $\nu_n = n$, Bohr's theorem holds: if f converges somewhere and has an analytic extension which is bounded in a half-plane $\{\operatorname{Re} s > \theta\}$, then it actually converges uniformly in every half-plane $\{\operatorname{Re} s > \theta + \varepsilon\}$, $\varepsilon > 0$. We prove, under very mild conditions, that given a sequence of Beurling primes, a small perturbation yields another sequence of primes such that the corresponding Beurling integers satisfy Bohr's condition, and therefore the theorem. Applying our result in conjunction with work of Diamond–Montgomery–Vorhauer and Zhang, we find a system of Beurling primes for which both Bohr's theorem and the Riemann hypothesis are valid. We will discuss the connections between our work with a conjecture of Helson concerning outer functions in Hardy spaces of generalized Dirichlet series.

The space $\mathcal{H}_q^2$ of generalized Dirichlet series with square summable coefficients is defined as

$$\mathcal{H}_q^2 = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{\nu_n^s} : \|f\|^2_{\mathcal{H}_q^2} = \sum_{n\geq 1} |a_n|^2 < +\infty \right\}.$$

In the classical case, the space $\mathcal{H}^2 = \left\{ f(s) = \sum_{n\geq 1} \frac{a_n}{n^s} : \|f\|^2 = \sum_{n\geq 1} |a_n|^2 < \infty \right\}$, was first systematically studied in an influential article of Hedenmalm, Lindqvist, and Seip.

Compact composition operators on $\mathcal{H}^2$

Gordon and Hedenmalm determined the class $\mathfrak{G}$ of analytic function $\psi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ that induce bounded composition operators $C_\psi(f) = f \circ \psi$ on $\mathcal{H}^2$, where by $\mathbb{C}_\theta$ we denote the half–plane $\{z : \operatorname{Re} z \geq \theta\}$, $\theta \in \mathbb{R}$.

O. F. Brevig and K–M. Perfekt characterized compact composition operators on $\mathcal{H}^2$, with Dirichlet series symbols. For the remaining class of symbols, F. Bayart gave a sufficient condition for the composition operator C_ψ to be compact. Our goal is to give a necessary condition in terms of the counting function

$$\mathcal{N}_\psi(w) = \sum_{\substack{s\in\psi^{-1}(\{w\}) \\ \operatorname{Re} s>0}} \operatorname{Re} s.$$

To do that we will add Beurling primes to the structure of $\mathcal{H}^2$ and prove that this does not have an effect on the behavior of such composition operators.