Local and global Sobolev regularity of quasiconformal maps

A. Walton Green (WUSTL)

with Francesco Di Plinio (UNINA) and Brett D. Wick (WUSTL)

November 6, 2023

WUSTL Analysis Seminar

Beltrami equation. For $K \geq 1$, a map $f : \mathbb{C} \to \mathbb{C}$ is K-quasiregular if $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ and satisfies the Beltrami equation

$$\bar{\partial} f = \mu \partial f, \qquad \|\mu\|_\infty = \frac{K-1}{K+1}. \tag{B}$$

If in addition, f is a homeomorphism, f is said to be K-quasiconformal.

Beltrami equation. For $K \geq 1$, a map $f : \mathbb{C} \to \mathbb{C}$ is K-quasiregular if $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ and satisfies the Beltrami equation

$$\bar{\partial} f = \mu \partial f, \qquad \|\mu\|_\infty = \frac{K-1}{K+1}. \tag{B}$$

If in addition, f is a homeomorphism, f is said to be K-quasiconformal.

1-quasiregular $\leftrightarrow$ analytic. 1-quasiconformal $\leftrightarrow$ conformal.

Beltrami equation. For $K \geq 1$, a map $f : \mathbb{C} \to \mathbb{C}$ is K-quasiregular if $f \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and satisfies the Beltrami equation

$$\bar{\partial} f = \mu \partial f, \qquad \|\mu\|_\infty = \frac{K-1}{K+1}. \tag{B}$$

If in addition, f is a homeomorphism, f is said to be K-quasiconformal.

1-quasiregular $\leftrightarrow$ analytic. 1-quasiconformal $\leftrightarrow$ conformal.

For any domain $\Omega \subseteq \mathbb{C}$, the Sobolev space $W^{n,p}(\Omega)$ is defined by the norm

$$\|g\|_{W^{n,p}(\Omega)} = \|g\|_{L^p(\Omega)} + \|\nabla^n g\|_{L^p(\Omega)}.$$

We say $f \in W^{n,p}_{\mathrm{loc}}(\Omega)$ if $\eta f \in W^{n,p}(\Omega)$ for every $\eta \in C_0^\infty(\Omega)$.

Beltrami equation. For $K \geq 1$, a map $f : \mathbb{C} \to \mathbb{C}$ is K-quasiregular if $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ and satisfies the Beltrami equation

$$\bar{\partial} f = \mu \partial f, \qquad \|\mu\|_\infty = \frac{K-1}{K+1}. \tag{B}$$

If in addition, f is a homeomorphism, f is said to be K-quasiconformal.

1-quasiregular $\leftrightarrow$ analytic. 1-quasiconformal $\leftrightarrow$ conformal.

Since $\frac{K-1}{K+1} < 1$, (B) is an elliptic PDE.

Beltrami equation. For $K \geq 1$, a map $f : \mathbb{C} \to \mathbb{C}$ is K-quasiregular if $f \in W^{1,2}_{\text{loc}}(\mathbb{C})$ and satisfies the Beltrami equation

$$\bar{\partial} f = \mu \partial f, \qquad \|\mu\|_\infty = \frac{K-1}{K+1}. \tag{B}$$

If in addition, f is a homeomorphism, f is said to be K-quasiconformal.

1-quasiregular $\leftrightarrow$ analytic. 1-quasiconformal $\leftrightarrow$ conformal.

Since $\frac{K-1}{K+1} < 1$, (B) is an elliptic PDE.

So, for any μ with $\|\mu\|_\infty < 1$, there exists a unique principal solution f which is quasiconformal and

$$f(z) = z + O(z^{-1}), \qquad |z| \to \infty.$$

Example: Radial stretching.

Let us demonstrate that $f(z) = z|z|^{\frac{1}{K}-1}$ is K-quasiconformal.

Example: Radial stretching.

Let us demonstrate that $f(z) = z|z|^{\frac{1}{K}-1}$ is K-quasiconformal.

Factoring $z|z|^{\frac{1}{K}-1} = zz^{\frac{1-K}{2K}}\bar{z}^{\frac{1-K}{2K}} = z^{\frac{1+K}{2K}}\bar{z}^{\frac{1-K}{2K}}$, we can calculate

$$\bar{\partial} f(z) = \frac{1-K}{2K}\frac{f(z)}{\bar{z}}, \qquad \partial f(z) = \frac{1+K}{2K}\frac{f(z)}{z}.$$

Example: Radial stretching.

Let us demonstrate that $f(z) = z|z|^{\frac{1}{K}-1}$ is K-quasiconformal.

Factoring $z|z|^{\frac{1}{K}-1} = zz^{\frac{1-K}{2K}}\bar{z}^{\frac{1-K}{2K}} = z^{\frac{1+K}{2K}}\bar{z}^{\frac{1-K}{2K}}$, we can calculate

$$\bar{\partial}f(z) = \frac{1-K}{2K}\frac{f(z)}{\bar{z}}, \qquad \partial f(z) = \frac{1+K}{2K}\frac{f(z)}{z}.$$

Therefore, $\bar{\partial}f = \mu\partial f$ with

$$\mu(z) = \frac{1-K}{1+K}\frac{z}{\bar{z}}, \qquad |\mu(z)| = \frac{K-1}{K+1}.$$

Example: Radial stretching.

Let us demonstrate that $f(z) = z|z|^{\frac{1}{K}-1}$ is K-quasiconformal.

Factoring $z|z|^{\frac{1}{K}-1} = zz^{\frac{1-K}{2K}}\bar{z}^{\frac{1-K}{2K}} = z^{\frac{1+K}{2K}}\bar{z}^{\frac{1-K}{2K}}$, we can calculate

$$\bar{\partial} f(z) = \frac{1-K}{2K}\frac{f(z)}{\bar{z}}, \qquad \partial f(z) = \frac{1+K}{2K}\frac{f(z)}{z}.$$

Therefore, $\bar{\partial} f = \mu \partial f$ with

$$\mu(z) = \frac{1-K}{1+K}\frac{z}{\bar{z}}, \qquad |\mu(z)| = \frac{K-1}{K+1}.$$

<u>Exercise</u>: Check that $f \in W^{1,q}_{\text{loc}}$ for $q < \frac{2K}{K-1}$.

Example: Radial stretching.

Let us demonstrate that $f(z) = z|z|^{\frac{1}{K}-1}$ is K-quasiconformal.

Factoring $z|z|^{\frac{1}{K}-1} = zz^{\frac{1-K}{2K}}\bar{z}^{\frac{1-K}{2K}} = z^{\frac{1+K}{2K}}\bar{z}^{\frac{1-K}{2K}}$, we can calculate

$$\bar{\partial} f(z) = \frac{1-K}{2K}\frac{f(z)}{\bar{z}}, \qquad \partial f(z) = \frac{1+K}{2K}\frac{f(z)}{z}.$$

Therefore, $\bar{\partial} f = \mu \partial f$ with

$$\mu(z) = \frac{1-K}{1+K}\frac{z}{\bar{z}}, \qquad |\mu(z)| = \frac{K-1}{K+1}.$$

<u>Exercise</u>: Check that $f \in W^{1,q}_{\mathrm{loc}}$ for $q < \frac{2K}{K-1}$. $|\nabla f(z)| \sim |z|^{\frac{1-K}{K}}$

Example: Radial stretching.

Let us demonstrate that $f(z) = z|z|^{\frac{1}{K}-1}$ is K-quasiconformal.

Factoring $z|z|^{\frac{1}{K}-1} = zz^{\frac{1-K}{2K}}\bar{z}^{\frac{1-K}{2K}} = z^{\frac{1+K}{2K}}\bar{z}^{\frac{1-K}{2K}}$, we can calculate

$$\bar{\partial} f(z) = \frac{1-K}{2K}\frac{f(z)}{\bar{z}}, \qquad \partial f(z) = \frac{1+K}{2K}\frac{f(z)}{z}.$$

Therefore, $\bar{\partial} f = \mu \partial f$ with

$$\mu(z) = \frac{1-K}{1+K}\frac{z}{\bar{z}}, \qquad |\mu(z)| = \frac{K-1}{K+1}.$$

Exercise: Check that $f \in W^{1,q}_{\text{loc}}$ for $q < \frac{2K}{K-1}$. $|\nabla f(z)| \sim |z|^{\frac{1-K}{K}}$

Exercise: What happens when $q = \frac{2K}{K-1}$?

Bojarski's Theorem on self-improving regularity (1955).

For each $K \geq 1$, there exists a critical interval (q_K, p_K) satisfying:

If for some $p \in (q_K, p_K)$, $f \in W^{1,p}_{\mathrm{loc}}$ satisfies

$$\bar{\partial} f = \mu \partial f, \quad \|\mu\|_\infty \leq \frac{K-1}{K+1}, \tag{B}$$

then $f \in W^{1,q}_{\mathrm{loc}}$ for all $q \in (q_K, p_K)$.

Bojarski's Theorem on self-improving regularity (1955).

For each $K \geq 1$, there exists a critical interval (q_K, p_K) satisfying:

If for <u>some</u> $p \in (q_K, p_K)$, $f \in W^{1,p}_{\mathrm{loc}}$ satisfies

$$\bar{\partial} f = \mu \partial f, \quad \|\mu\|_\infty \leq \frac{K-1}{K+1}, \tag{B}$$

then $f \in W^{1,q}_{\mathrm{loc}}$ for <u>all</u> $q \in (q_K, p_K)$.

The proof relies on the Beurling transform T defined by

$$Tg(z) = \lim_{\varepsilon \to 0} \int_{|z-w|>\varepsilon} \frac{g(w)}{(z-w)^2} \, dw.$$

Beurling Transform.

T is a Calderón-Zygmund operator (CZO), so T is bounded on L^p for $1 < p < \infty$.

Beurling Transform.

T is a Calderón-Zygmund operator (CZO), so T is bounded on L^p for $1 < p < \infty$.

Intertwining property: if $g \in W^{1,2}$, then $T(\bar{\partial} g) = \partial g$.

Beurling Transform.

T is a Calderón-Zygmund operator (CZO), so T is bounded on L^p for $1 < p < \infty$.

Intertwining property: if $g \in W^{1,2}$, then $T(\bar{\partial} g) = \partial g$.

Let $f \in W^{1,2}_{loc}$ satisfy (B), let $\eta \in C_0^\infty$, and set $w = \eta f \in W^{1,2}$.

$$\bar{\partial} w = \mu \partial w + (\bar{\partial}\eta + \mu \partial \eta) f = \mu T(\bar{\partial} w) + h.$$

$$\bar{\partial} w = (I - \mu T)^{-1} h, \quad \partial w = T(I - \mu T)^{-1} h.$$

Beurling Transform.

T is a Calderón-Zygmund operator (CZO), so T is bounded on L^p for $1 < p < \infty$.

Intertwining property: if $g \in W^{1,2}$, then $T(\bar{\partial} g) = \partial g$.

Let $f \in W^{1,2}_{\text{loc}}$ satisfy (B), let $\eta \in C_0^\infty$, and set $w = \eta f \in W^{1,2}$.

$$\bar{\partial} w = \mu \partial w + (\bar{\partial}\eta + \mu\partial\eta) f = \mu T(\bar{\partial} w) + h.$$

$$\bar{\partial} w = (I - \mu T)^{-1} h, \quad \partial w = T(I - \mu T)^{-1} h.$$

Principle: Local regularity of f follows from global regularity of w satisfying the inhomogeneous Beltrami equation

$$\bar{\partial} w = \mu \partial w + h,$$

or equivalently, norm estimates on the Beltrami resolvent $(I - \mu T)^{-1}$.

Caccioppoli inequalities. If $I - \mu T$ is invertible on L^p, then for all $\eta \in C_0^\infty$ and f satisfying (B),

$$\|\eta(\nabla f)\|_{L^p} \lesssim \|(\nabla \eta) f\|_{L^p}.$$

Caccioppoli inequalities. If $I - \mu T$ is invertible on L^p, then for all $\eta \in C_0^\infty$ and f satisfying (B),

$$\|\eta(\nabla f)\|_{L^p} \lesssim \|(\nabla \eta) f\|_{L^p}.$$

An iteration procedure implies Bojarski's theorem.

Caccioppoli inequalities. If $I - \mu T$ is invertible on L^p, then for all $\eta \in C_0^\infty$ and f satisfying (B),

$$\|\eta(\nabla f)\|_{L^p} \lesssim \|(\nabla \eta) f\|_{L^p}.$$

An iteration procedure implies Bojarski's theorem.

Theorem. [Astala–Iwaniec–Saksman '01]
If $\frac{2K}{K+1} < p < \frac{2K}{K-1}$, then $I - \mu T$ is invertible on L^p.

Caccioppoli inequalities. If $I - \mu T$ is invertible on L^p, then for all $\eta \in C_0^\infty$ and f satisfying (B),

$$\|\eta(\nabla f)\|_{L^p} \lesssim \|(\nabla \eta) f\|_{L^p}.$$

An iteration procedure implies Bojarski's theorem.

Theorem. [Astala–Iwaniec–Saksman '01]
If $\frac{2K}{K+1} < p < \frac{2K}{K-1}$, then $I - \mu T$ is invertible on L^p.

Key ingredients in proof:

1. Area Distortion [Astala '94]: For the principal solution f, the Jacobian $|Jf^{-1}|^{1-p/2}$ belongs to the Muckenhoupt weight class A_p.

Caccioppoli inequalities. If $I - \mu T$ is invertible on L^p, then for all $\eta \in C_0^\infty$ and f satisfying (B),

$$\|\eta(\nabla f)\|_{L^p} \lesssim \|(\nabla \eta) f\|_{L^p}.$$

An iteration procedure implies Bojarski's theorem.

Theorem. [Astala–Iwaniec–Saksman '01]
If $\frac{2K}{K+1} < p < \frac{2K}{K-1}$, then $I - \mu T$ is invertible on L^p.

Key ingredients in proof:

1. Area Distortion [Astala '94]: For the principal solution f, the Jacobian $|Jf^{-1}|^{1-p/2}$ belongs to the Muckenhoupt weight class A_p.

2. CZOs are bounded on the weighted Lebesgue space $L^p(\omega)$ for ω in A_p [Coifman-Fefferman '74].

Caccioppoli inequalities. If $I - \mu T$ is invertible on L^p, then for all $\eta \in C_0^\infty$ and f satisfying (B),

$$\|\eta(\nabla f)\|_{L^p} \lesssim \|(\nabla \eta) f\|_{L^p}.$$

An iteration procedure implies Bojarski's theorem.

Theorem. [Astala–Iwaniec–Saksman '01]
If $\frac{2K}{K+1} < p < \frac{2K}{K-1}$, then $I - \mu T$ is invertible on L^p.

Key ingredients in proof:

1. Area Distortion [Astala '94]: For the principal solution f, the Jacobian $|Jf^{-1}|^{1-p/2}$ belongs to the Muckenhoupt weight class A_p.

2. CZOs are bounded on the weighted Lebesgue space $L^p(\omega)$ for ω in A_p [Coifman-Fefferman '74].

Invertiblity fails at the endpoints by considering radial stretchings.

Astala-Iwnaiec-Saksman Strategy. It suffices to prove

$$\|\bar{\partial} w\|_{L^p} \lesssim \|h\|_{L^p}, \quad \text{for} \quad \bar{\partial} w = \mu \partial w + h,$$

for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$.

Astala-Iwnaiec-Saksman Strategy. It suffices to prove

$$\|\bar\partial w\|_{L^p} \lesssim \|h\|_{L^p}, \quad \text{for} \quad \bar\partial w = \mu \partial w + h,$$

for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Introducing $u = w \circ f^{-1}$ where f is the principal solution to $\bar\partial f = \mu \partial f$,

$$\bar\partial w = (\partial u \circ f)\bar\partial f + (\bar\partial u \circ f)\overline{\partial f}.$$

$$(\bar\partial u \circ f)\overline{\partial f} = \frac{h}{1-|\mu|^2} \in L^p. \tag{H}$$

Astala-Iwnaiec-Saksman Strategy. It suffices to prove

$$\|\bar\partial w\|_{L^p} \lesssim \|h\|_{L^p}, \quad \text{for} \quad \bar\partial w = \mu \partial w + h,$$

for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Introducing $u = w \circ f^{-1}$ where f is the principal solution to $\bar\partial f = \mu \partial f$,

$$\bar\partial w = (\partial u \circ f)\bar\partial f + (\bar\partial u \circ f)\overline{\partial f}.$$

$$(\bar\partial u \circ f)\overline{\partial f} = \frac{h}{1-|\mu|^2} \in L^p. \tag{H}$$

By change of variable, and the intertwining property of T,

$$\|\bar\partial w\|_{L^p}^p \lesssim \|h\|_{L^p}^p + \int |T(\bar\partial u)|^p |Jf^{-1}|^{1-p/2},$$

Astala-Iwnaiec-Saksman Strategy. It suffices to prove

$$\|\bar\partial w\|_{L^p} \lesssim \|h\|_{L^p}, \quad \text{for} \quad \bar\partial w = \mu \partial w + h,$$

for $\frac{2K}{K+1} < p < \frac{2K}{K-1}$. Introducing $u = w \circ f^{-1}$ where f is the principal solution to $\bar\partial f = \mu \partial f$,

$$\bar\partial w = (\partial u \circ f)\bar\partial f + (\bar\partial u \circ f)\overline{\partial f}.$$

$$(\bar\partial u \circ f)\overline{\partial f} = \frac{h}{1-|\mu|^2} \in L^p. \tag{H}$$

By change of variable, and the intertwining property of T,

$$\|\bar\partial w\|_{L^p}^p \lesssim \|h\|_{L^p}^p + \int |T(\bar\partial u)|^p |Jf^{-1}|^{1-p/2},$$

so the result follows from 1. and 2. and changing variables back.

Sobolev regularity of μ. Suppose $\nabla\mu \in L^p(\mathbb{C})$, $p > 2$.

Sobolev regularity of μ. Suppose $\nabla\mu \in L^p(\mathbb{C})$, $p > 2$.

If $f \in L^{\frac{p}{p-1}}_{\text{loc}}$ and satisfies the distributional Beltrami equation

$$\langle f, \bar{\partial}\phi - \partial(\mu\phi)\rangle = 0 \qquad \phi \in C_0^\infty,$$

then [Clop et. al. '09] showed $f \in W^{2,p}_{\text{loc}}$.

Sobolev regularity of μ. Suppose $\nabla\mu \in L^p(\mathbb{C})$, $p > 2$.

If $f \in L^{\frac{p}{p-1}}_{\text{loc}}$ and satisfies the distributional Beltrami equation

$$\langle f, \bar{\partial}\phi - \partial(\mu\phi)\rangle = 0 \qquad \phi \in C_0^\infty,$$

then [Clop et. al. '09] showed $f \in W^{2,p}_{\text{loc}}$.

In this case, not only do we gain in regularity, from p to $\frac{p}{p-1}$, but we also also gain two derivatives!

Sobolev regularity of μ. Suppose $\nabla\mu \in L^p(\mathbb{C})$, $p > 2$.

If $f \in L^{\frac{p}{p-1}}_{\text{loc}}$ and satisfies the distributional Beltrami equation

$$\langle f, \bar{\partial}\phi - \partial(\mu\phi)\rangle = 0 \qquad \phi \in C_0^\infty,$$

then [Clop et. al. '09] showed $f \in W^{2,p}_{\text{loc}}$.

In this case, not only do we gain in regularity, from p to $\frac{p}{p-1}$, but we also also gain two derivatives!

Method: Stoilow factorization and Weyl Lemma (holomorphic distributions are holomorphic functions a.e.)

Beltrami resolvent on smoothness spaces.

Later, [Mateu-Orobitg-Verdera '09] and [Cruz-M.-O. '13] studied invertibility of $I - \mu T$ on various supercritical smoothness spaces, following [T. Iwaniec '92] method for $\mu \in \mathrm{VMO}$.

$$\left(\sum_{n=0}^{N-1} (\mu T)^n\right)(I - \mu T) = I - (\mu T)^N$$

$$= I - \mu^N T^N + \mu^N T^N - (\mu T)^N.$$

Beltrami resolvent on smoothness spaces.

Later, [Mateu-Orobitg-Verdera '09] and [Cruz-M.-O. '13] studied invertibility of $I - \mu T$ on various supercritical smoothness spaces, following [T. Iwaniec '92] method for $\mu \in \mathrm{VMO}$.

$$\left(\sum_{n=0}^{N-1} (\mu T)^n\right)(I - \mu T) = I - (\mu T)^N$$
$$= I - \mu^N T^N + \mu^N T^N - (\mu T)^N.$$

Since $\|\mu\|_\infty < 1$ and $\|T^N\| \lesssim N^\alpha$, $I - \mu^N T^N$ is invertible for large N.

Beltrami resolvent on smoothness spaces.

Later, [Mateu-Orobitg-Verdera '09] and [Cruz-M.-O. '13] studied invertibility of $I - \mu T$ on various supercritical smoothness spaces, following [T. Iwaniec '92] method for $\mu \in \mathrm{VMO}$.

$$\left(\sum_{n=0}^{N-1} (\mu T)^n\right)(I - \mu T) = I - (\mu T)^N$$

$$= I - \mu^N T^N + \mu^N T^N - (\mu T)^N.$$

Since $\|\mu\|_\infty < 1$ and $\|T^N\| \lesssim N^\alpha$, $I - \mu^N T^N$ is invertible for large N.

$\mu^N T^N - (\mu T)^N$ is compact since $[\mu, T]$ is compact.

Beltrami resolvent on smoothness spaces.

Later, [Mateu-Orobitg-Verdera '09] and [Cruz-M.-O. '13] studied invertibility of $I - \mu T$ on various supercritical smoothness spaces, following [T. Iwaniec '92] method for $\mu \in \mathrm{VMO}$.

$$\left(\sum_{n=0}^{N-1} (\mu T)^n\right)(I - \mu T) = I - (\mu T)^N$$

$$= I - \mu^N T^N + \mu^N T^N - (\mu T)^N.$$

Since $\|\mu\|_\infty < 1$ and $\|T^N\| \lesssim N^\alpha$, $I - \mu^N T^N$ is invertible for large N.

$\mu^N T^N - (\mu T)^N$ is compact since $[\mu, T]$ is compact.

Conclude with injectivity of $I - \mu T$ which follows from [Iwaniec '92].

Compressed Beltrami resolvent. These results also apply to

$$(I_\Omega - \mu T_\Omega)^{-1}, \qquad S_\Omega := \mathbf{1}_{\overline{\Omega}} S(\mathbf{1}_{\overline{\Omega}} \cdot),$$

where $\Omega \subseteq \mathbb{C}$ is a bounded Lipschitz domain and $\mu \in W^{1,p}(\Omega)$.

Compressed Beltrami resolvent. These results also apply to

$$(I_\Omega - \mu T_\Omega)^{-1}, \qquad S_\Omega := \mathbf{1}_{\overline{\Omega}} S(\mathbf{1}_{\overline{\Omega}} \cdot),$$

where $\Omega \subseteq \mathbb{C}$ is a bounded Lipschitz domain and $\mu \in W^{1,p}(\Omega)$.

Main Obstacle: estimating $\|T_\Omega^N\|_{W^{1,p}(\Omega)}$.

Compressed Beltrami resolvent. These results also apply to

$$(I_\Omega - \mu T_\Omega)^{-1}, \qquad S_\Omega := \mathbf{1}_{\overline{\Omega}} S(\mathbf{1}_{\overline{\Omega}} \cdot),$$

where $\Omega \subseteq \mathbb{C}$ is a bounded Lipschitz domain and $\mu \in W^{1,p}(\Omega)$.

Main Obstacle: estimating $\|T_\Omega^N\|_{W^{1,p}(\Omega)}$.

Suppose $S : W^{1,p}(\mathbb{C}) \to W^{1,p}(\mathbb{C})$. Does $S_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$?

Compressed Beltrami resolvent. These results also apply to

$$(I_\Omega - \mu T_\Omega)^{-1}, \qquad S_\Omega := \mathbf{1}_{\overline{\Omega}} S(\mathbf{1}_{\overline{\Omega}} \cdot),$$

where $\Omega \subseteq \mathbb{C}$ is a bounded Lipschitz domain and $\mu \in W^{1,p}(\Omega)$.

Main Obstacle: estimating $\|T_\Omega^N\|_{W^{1,p}(\Omega)}$.

Suppose $S : W^{1,p}(\mathbb{C}) \to W^{1,p}(\mathbb{C})$. Does $S_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$?

Thereom. [Prats-Tolsa '15]
If S as above is a CZO, and $p > 2$, then

$$S_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega) \iff S_\Omega(\mathbf{1}) \in W^{1,p}(\Omega).$$

Compressed Beltrami resolvent. These results also apply to

$$(I_\Omega - \mu T_\Omega)^{-1}, \qquad S_\Omega := \mathbf{1}_{\overline{\Omega}} S(\mathbf{1}_{\overline{\Omega}} \cdot),$$

where $\Omega \subseteq \mathbb{C}$ is a bounded Lipschitz domain and $\mu \in W^{1,p}(\Omega)$.

Main Obstacle: estimating $\|T_\Omega^N\|_{W^{1,p}(\Omega)}$.

Suppose $S : W^{1,p}(\mathbb{C}) \to W^{1,p}(\mathbb{C})$. Does $S_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$?

Thereom. [Prats-Tolsa '15]
If S as above is a CZO, and $p > 2$, then

$$S_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega) \iff S_\Omega(\mathbf{1}) \in W^{1,p}(\Omega).$$

Furthermore, [Cruz-Tolsa '13 $N = 1$, Prats '17 $N > 1$] showed

$$\|T_\Omega^N(1)\|_{W^{1,p}(\Omega)} \lesssim_N 1 + \|\nu_\Omega\|_{B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)}.$$

Quantitative Sobolev regularity on $\mathbb{C}$.

Thereom. [Di Plinio, G., Wick '23]
For $\mu \in W^{1,2}(\mathbb{C})$, with $\|\mu\|_\infty = \frac{K-1}{K+1}$,

$$\left\|(I-\mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim \exp(K, p, \|\mu\|_{W^{1,2}}), \quad 1 < p < 2.$$

If in addition $\mu \in W^{1,p}(\mathbb{C})$ for some $p > 2$,

$$\left\|(I-\mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim 1 + \|\mu\|^2_{W^{1,p}(\mathbb{C})}.$$

Quantitative Sobolev regularity on $\mathbb{C}$.

Thereom. [Di Plinio, G., Wick '23]
For $\mu \in W^{1,2}(\mathbb{C})$, with $\|\mu\|_\infty = \frac{K-1}{K+1}$,

$$\left\|(I-\mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim \exp(K, p, \|\mu\|_{W^{1,2}}), \quad 1 < p < 2.$$

If in addition $\mu \in W^{1,p}(\mathbb{C})$ for some $p > 2$,

$$\left\|(I-\mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim 1 + \|\mu\|^2_{W^{1,p}(\mathbb{C})}.$$

Extend Astala-Iwaniec-Saksman strategy to Sobolev spaces.

Quantitative Sobolev regularity on $\mathbb{C}$.

Thereom. [Di Plinio, G., Wick '23]
For $\mu \in W^{1,2}(\mathbb{C})$, with $\|\mu\|_\infty = \frac{K-1}{K+1}$,

$$\left\|(I - \mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim \exp(K, p, \|\mu\|_{W^{1,2}}), \quad 1 < p < 2.$$

If in addition $\mu \in W^{1,p}(\mathbb{C})$ for some $p > 2$,

$$\left\|(I - \mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim 1 + \|\mu\|^2_{W^{1,p}(\mathbb{C})}.$$

Extend Astala-Iwaniec-Saksman strategy to Sobolev spaces. Since T is of convolution type, $\|T\|_{W^{1,p}(\mathbb{C},\omega)} \sim \|T\|_{L^p(\mathbb{C},\omega)}$.

Quantitative Sobolev regularity on $\mathbb{C}$.

Thereom. [Di Plinio, G., Wick '23]
For $\mu \in W^{1,2}(\mathbb{C})$, with $\|\mu\|_\infty = \frac{K-1}{K+1}$,

$$\left\|(I-\mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim \exp(K, p, \|\mu\|_{W^{1,2}}), \quad 1 < p < 2.$$

If in addition $\mu \in W^{1,p}(\mathbb{C})$ for some $p > 2$,

$$\left\|(I-\mu T)^{-1}\right\|_{\mathcal{L}(W^{1,p}(\mathbb{C}))} \lesssim 1 + \|\mu\|^2_{W^{1,p}(\mathbb{C})}.$$

Extend Astala-Iwaniec-Saksman strategy to Sobolev spaces. Since T is of convolution type, $\|T\|_{W^{1,p}(\mathbb{C},\omega)} \sim \|T\|_{L^p(\mathbb{C},\omega)}$.

The main novel ingredient is for the principal solution f and $\mu \in W^{1,2}$,

$$[\omega]_{A_p(\mathbb{C})} = \sup_{Q \text{ cube}} \langle \omega \rangle_Q \langle \omega^{\frac{-1}{p-1}} \rangle_Q^{p-1} < \infty, \quad \omega = |Jf^{-1}|^a,\ a \in \mathbb{R}.$$

Quantitative Sobolev regularity on Ω.

The compressed resolvent connects to the principal solution f of

$$\bar{\partial} f = (\mathbf{1}_{\overline{\Omega}}\mu)\,\partial f, \quad \mu \in W^{1,p}(\Omega)$$

through the formula

$$\bar{\partial} f = (I_\Omega - \mu T_\Omega)^{-1}\mu.$$

Quantitative Sobolev regularity on Ω.

The compressed resolvent connects to the principal solution f of

$$\bar{\partial} f = (\mathbf{1}_{\overline{\Omega}}\mu)\,\partial f, \quad \mu \in W^{1,p}(\Omega)$$

through the formula

$$\bar{\partial} f = (I_\Omega - \mu T_\Omega)^{-1}\mu.$$

So, if $I_\Omega - \mu T_\Omega$ is invertible on $W^{1,p}(\Omega)$ and $\nu_\Omega \in B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)$,

Quantitative Sobolev regularity on Ω.

The compressed resolvent connects to the principal solution f of

$$\bar{\partial} f = (\mathbf{1}_{\overline{\Omega}}\mu)\,\partial f, \quad \mu \in W^{1,p}(\Omega)$$

through the formula

$$\bar{\partial} f = (I_\Omega - \mu T_\Omega)^{-1}\mu.$$

So, if $I_\Omega - \mu T_\Omega$ is invertible on $W^{1,p}(\Omega)$ and $\nu_\Omega \in B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)$, then $O = f(\Omega)$ also satisfies $\nu_O \in B^{1-\frac{1}{p}}_{p,p}(\partial O)$.

Quantitative Sobolev regularity on Ω.

The compressed resolvent connects to the principal solution f of

$$\bar{\partial} f = (\mathbf{1}_{\overline{\Omega}} \mu)\, \partial f, \quad \mu \in W^{1,p}(\Omega)$$

through the formula

$$\bar{\partial} f = (I_\Omega - \mu T_\Omega)^{-1} \mu.$$

So, if $I_\Omega - \mu T_\Omega$ is invertible on $W^{1,p}(\Omega)$ and $\nu_\Omega \in B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)$, then $O = f(\Omega)$ also satisfies $\nu_O \in B^{1-\frac{1}{p}}_{p,p}(\partial O)$. Because the trace of $W^{2,p}$ is $B^{1-\frac{1}{p}}_{p,p}$.

Quantitative Sobolev regularity on Ω.

The compressed resolvent connects to the principal solution f of

$$\bar{\partial} f = (\mathbf{1}_{\overline{\Omega}}\mu)\,\partial f, \quad \mu \in W^{1,p}(\Omega)$$

through the formula

$$\bar{\partial} f = (I_\Omega - \mu T_\Omega)^{-1}\mu.$$

So, if $I_\Omega - \mu T_\Omega$ is invertible on $W^{1,p}(\Omega)$ and $\nu_\Omega \in B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)$, then $O = f(\Omega)$ also satisfies $\nu_O \in B^{1-\frac{1}{p}}_{p,p}(\partial O)$. Because the trace of $W^{2,p}$ is $B^{1-\frac{1}{p}}_{p,p}$.

Therefore, we introduce the sharp quantitative measurement of boundary regularity

$$\mathscr{O} = 1 + \|\nu_\Omega\|_{B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)} + \|\nu_O\|_{B^{1-\frac{1}{p}}_{p,p}(\partial O)}.$$

Quantitative Sobolev regularity on Ω. (continued)

$$\mathscr{O} = 1 + \|\nu_\Omega\|_{B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)} + \|\nu_O\|_{B^{1-\frac{1}{p}}_{p,p}(\partial O)}, \qquad O = f(\Omega).$$

Theorem. [Di Plinio, G., Wick, Fall '23]
If $\mu \in W^{1,p}(\Omega)$ for some $p > 2$ and Ω is simply connected,

$$\|(I - \mu T_\Omega)^{-1}\|_{\mathcal{L}(W^{1,p}(\Omega))} \lesssim \mathscr{O}\, \|T_O\|_{\mathcal{L}(W^{1,p}(O,\omega))}$$
$$\omega = |Jf^{-1}|^{1-p} \qquad\qquad + \mathscr{O}^4 \left(1 + \|\mu\|^6_{W^{1,p}(\Omega)}\right).$$

Quantitative Sobolev regularity on Ω. (continued)

$$\mathscr{O} = 1 + \|\nu_\Omega\|_{B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)} + \|\nu_O\|_{B^{1-\frac{1}{p}}_{p,p}(\partial O)}, \qquad O = f(\Omega).$$

Theorem. [Di Plinio, G., Wick, Fall '23]
If $\mu \in W^{1,p}(\Omega)$ for some $p > 2$ and Ω is simply connected,

$$\|(I - \mu T_\Omega)^{-1}\|_{\mathcal{L}(W^{1,p}(\Omega))} \lesssim \mathscr{O}\, \|T_O\|_{\mathcal{L}(W^{1,p}(O,\omega))}$$

$$\omega = |Jf^{-1}|^{1-p} \qquad\qquad + \mathscr{O}^4 \left(1 + \|\mu\|^6_{W^{1,p}(\Omega)}\right).$$

How can we tell that T_O is bounded on the weighted Sobolev space $W^{1,p}(O, \omega)$?

Quantitative Sobolev regularity on Ω. (continued)

$$\mathscr{O} = 1 + \|\nu_\Omega\|_{B^{1-\frac{1}{p}}_{p,p}(\partial\Omega)} + \|\nu_O\|_{B^{1-\frac{1}{p}}_{p,p}(\partial O)}, \qquad O = f(\Omega).$$

Theorem. [Di Plinio, G., Wick, Fall '23]
If $\mu \in W^{1,p}(\Omega)$ for some $p > 2$ and Ω is simply connected,

$$\|(I - \mu T_\Omega)^{-1}\|_{\mathcal{L}(W^{1,p}(\Omega))} \lesssim \mathscr{O}\,\|T_O\|_{\mathcal{L}(W^{1,p}(O,\omega))} + \mathscr{O}^4\left(1 + \|\mu\|^6_{W^{1,p}(\Omega)}\right).$$

$$\omega = |Jf^{-1}|^{1-p}$$

How can we tell that T_O is bounded on the weighted Sobolev space $W^{1,p}(O,\omega)$?

We need to extend [Prats-Tolsa '15] to the weighted setting

Weighted $\nabla S_O(1)$-type Theorem.

Thereom. [Di Plinio, G., Wick, Spring '23]
If S is a smooth CZO, $p > 2$, and $\omega \in A_{\frac{p}{2}}(\Omega)$, then

$$\|S_O\|_{\mathcal{L}(W^{1,p}(O,\omega))} \lesssim 1 + \frac{\|S_O(\mathbf{1})\|_{W^{1,p}(O,\omega)}}{\|\mathbf{1}_O\|_{L^p(O,\omega)}} \lesssim 1 + \|S_O(\mathbf{1})\|_{W^{1,p+\varepsilon}(O)} .$$

Weighted $\nabla S_O(1)$-type Theorem.

Thereom. [Di Plinio, G., Wick, Spring '23]
If S is a smooth CZO, $p > 2$, and $\omega \in A_{\frac{p}{2}}(\Omega)$, then

$$\|S_O\|_{\mathcal{L}(W^{1,p}(O,\omega))} \lesssim 1 + \frac{\|S_O(\mathbf{1})\|_{W^{1,p}(O,\omega)}}{\|\mathbf{1}_O\|_{L^p(O,\omega)}} \lesssim 1 + \|S_O(\mathbf{1})\|_{W^{1,p+\varepsilon}(O)}.$$

Combining the two results, we obtain

Corollary. [Di Plinio, G. Wick, Fall '23] For any $2 < p < q$,

$$\|(I - \mu T_\Omega)^{-1}\|_{\mathcal{L}(W^{1,p}(\Omega))}$$

$$\lesssim \mathscr{O}\,\|\nu_O\|_{B_{q,q}^{1-\frac{1}{q}}(\partial O)} + \mathscr{O}^4\left(1 + \|\mu\|^6_{W^{1,p}(\Omega)}\right).$$

Weighted $\nabla S_O(1)$-type Theorem.

Thereom. [Di Plinio, G., Wick, Spring '23]
If S is a smooth CZO, $p > 2$, and $\omega \in A_{\frac{p}{2}}(\Omega)$, then

$$\|S_O\|_{\mathcal{L}(W^{1,p}(O,\omega))} \lesssim 1 + \frac{\|S_O(\mathbf{1})\|_{W^{1,p}(O,\omega)}}{\|\mathbf{1}_O\|_{L^p(O,\omega)}} \lesssim 1 + \|S_O(\mathbf{1})\|_{W^{1,p+\varepsilon}(O)}.$$

Combining the two results, we obtain

Corollary. [Di Plinio, G. Wick, Fall '23] For any $2 < p < q$,

$$\|(I - \mu T_\Omega)^{-1}\|_{\mathcal{L}(W^{1,p}(\Omega))}$$

$$\lesssim \mathscr{O}\,\|\nu_O\|_{B_{q,q}^{1-\frac{1}{q}}(\partial O)} + \mathscr{O}^4\left(1 + \|\mu\|^6_{W^{1,p}(\Omega)}\right).$$

Thank for your attention.