Sobolev Regularity of Singular Integral Operators on Domains

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Motivation: Beltrami equation.

For $K \geq 1$, a map $f : \mathbb{C} \to \mathbb{C}$ is K-quasiregular if $f \in W^{1,2}_{\text{loc}}$ and

$$f_{\bar{z}} = \mu f_z, \quad \|\mu\|_\infty = k = \frac{K-1}{K+1} < 1. \tag{B}$$

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The Measurable Riemann Mapping Theorem [Morrey '38] ensures the existence of quasiconformal f satisfying (B) for any $\|\mu\|_\infty < 1$.

Example: Radial stretching.

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Exercise: What happens when $q = \frac{2K}{K-1}$?

Bojarski's Theorem on self-improving regularity '55.

For each $K \geq 1$, there exists a critical interval (q_K, p_K) satisfying:

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Key property: $Tg_{\bar{z}} = g_z$.

Proof of Bojarski's Theorem. Set $w = \eta f$, $\eta \in C_0^\infty$. WTS $w \in W^{1,q}$

$$w_{\bar{z}} = \mu w_z + (\eta_{\bar{z}} + \mu \eta_z) f = \mu T w_{\bar{z}} + h.$$

$$w_{\bar{z}} = (I - \mu T)^{-1} h, \quad w_z = T(I - \mu T)^{-1} h.$$

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Using the radial stretchings $f(z) = z|z|^{\frac{1}{K} - 1}$, the largest the critical interval could be is

$$\left(\frac{2K}{K+1}, \frac{2K}{K-1} \right) = \left(1 + k, 1 + \frac{1}{k} \right).$$

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[Bañuelos–G. Wang '95] $c_p \leq 4$, [Nazarov–Volberg '03, Bañ.–Méndez-Hernández] $c_p \leq 2$, [Dragicevic–Vol. '05] $c_p \to 1$ as $p \to \infty$, [Bañ.–Janakiraman '08] $c_p \leq 1.575$.

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1. Area Distortion [Astala '94]: for any K-quasiconformal f, the Jacobian $|J_{f^{-1}}|^{1-p/2}$ belongs to the Muckenhoupt class A_p exactly for p in (C).

$$[\omega]_{A_p} = \sup_{B \text{ ball}} \left(\frac{1}{|B|}\int_B \omega\right)\left(\frac{1}{|B|}\int_B \omega^{1-p'}\right)^{p-1} < \infty.$$

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2. CZOs are bounded on $L^p(\omega)$ for ω in A_p [Coifman-Fefferman '74].

Astala-Iwnaiec-Saksman Strategy. It suffices to prove, for p in (C)

$$\|w_{\bar{z}}\|_{L^p} \lesssim \|h\|_{L^p}, \quad \text{for} \quad w_{\bar{z}} = \mu w_z + h.$$

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Conclude by changing variables back and applying (H).

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In this case, it is not obvious how to get even the unweighted bounds

$$\|T_\Omega f\|_{W^{n,p}(\Omega)} \lesssim \|f\|_{W^{n,p}(\Omega)}.$$

Various sufficient testing conditions of $T1$-type: [Mateu-Orobitg-Verdera, JMPA '09][Cruz-M.-O., CJM '13][Prats-Tolsa, JFA '15][P., Pub. Mat. '15]

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$$f = \sum_{Q \in \mathcal{W}} |Q| \langle f, \phi_Q \rangle \phi_Q + \sum_{R \in \mathcal{D}(\mathcal{W})} |R| \langle f, \psi_R \rangle \psi_R.$$

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$\mathcal{D}(\mathcal{W})$ are the dyadic subcubes of $\mathcal{W}$

$\mathcal{M} = \mathcal{W} \cup \mathcal{D}(\mathcal{W})$.

Singular Integral Operators (SIOs) on Ω. (n fixed)

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$$\langle Tf, g \rangle = \iint_{\Omega \times \Omega} K(x, y) f(y) g(x) \, dy \, dx,$$

where for each $j = 0, 1, \ldots, n+1$,

$$|\nabla^j K(x, y)| \lesssim \frac{1}{|x - y|^{d+j}}.$$

Representation Theorem. Let T be a smooth SIO on a Lipschitz domain Ω. Suppose that $T(x^\gamma) = 0$ for $|\gamma| \leq n - 1$. Then,

$$\langle \nabla^n T f, g \rangle = \sum_{Q \in \mathcal{W}} \langle \nabla^n f, \chi_Q \rangle \langle \theta_Q, g \rangle + \sum_{R \in \mathcal{D}(\mathcal{W})} \langle \nabla^n f, \varphi_R \rangle \langle \zeta_R, g \rangle + \text{paraproducts}$$

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These are classical paraproducts acting on $\nabla^n f, g$ with symbols $\nabla^n T(x^\gamma)$, $|\gamma| = n$ and $(\nabla^n T)^\star \mathbf{1}$, so for these operators we get a full sparse/weighted $T1$-type theorem.

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Key Property: $\Pi^\gamma_{\mathbf{b}}(x^\gamma, g) = \gamma! \langle \mathbf{b}, g \rangle$

Testing conditions.

Prop. Let $m = n - |\gamma|$.

$$\left\| \Pi_{\mathbf{b}}^{\gamma} : W^{m,p}(\Omega) \times L^{p'}(\Omega) \right\| \lesssim \|\mathbf{b}\|_{\mathrm{bmo}^{-m,p+}} + \|\mathbf{b}\|_{\mathrm{Carl}^{m,p}(\Omega)} .$$

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Prop. gives Sobolev bounds for T.

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Thank you for your attention.

Carleson and bmo norms.

Let $m \geq 1$, and $1 < p < \infty$.

$$\|f\|_{\mathrm{bmo}^{-m,p}} = \sup_{Q \in \mathcal{M}} \ell(Q)^m \left(\frac{1}{|Q|} \int_Q |f - f_Q|^p \right)^{1/p}, \quad f_Q = \frac{1}{|Q|} \int_Q f.$$

Let $\|f\|_{\mathrm{Carl}^{m,p}(\Omega)}$ denote the smallest C such that

$$\sup_{W \in \mathcal{W}} \ell(W)^m \langle f \rangle_{p,W} \leq C, \qquad \text{and}$$

$$\text{if } mp \leq d \qquad \forall \ell(W) \lesssim 1 \quad \sum_{P \in \mathrm{Sh}(W)} \|f\|_{L^p(\mathrm{Sh}(P))}^{pp'} \ell(P)^{\frac{mp-d}{p-1}} \leq C^{p'} \|f\|_{L^p(\mathrm{Sh}(W))}^p;$$

$$\text{if } mp > d \qquad \forall \ell(W) \sim 1, \quad \|f\|_{L^p(\mathrm{Sh}(W))} \leq C. \qquad \mathrm{Sh}(W) \sim 50M \cdot W$$