Algebra I, Fall 2016

Solutions to Problem Set 1

3. Let a_iK, $i \in I$, be all the distinct left cosets of K in G, and let $b_jH, j \in J, b_j \in K$ be all the left cosets of H in K. We show the $a_ib_jH, i \in I, j \in J$ are all the distinct left cosets of H in G.

First, if $a_ib_jH = a_{i'}b_{j'}H$, then $b_j^{-1}a_i^{-1}a_{i'}b_{j'} \in H \subset K$, so

$$a_ib_jK = a_{i'}b_{j'}K.$$

But $b_j, b_{j'} \in K$, so $b_jK = b_{j'}K = K$, so the above equality gives $a_iK = a_{i'}K$, therefore $i = i'$. And since $a_ib_jH = a_{i'}b_{j'}H$, we have $b_jH = b_{j'}H$, so $j = j'$.

Second, if xH is a left cost of H in G, then $xK = a_iK$ for some i, so $a_i^{-1}x \in K$, so $a_i^{-1}xH = b_jH$ for some j, so $b_j^{-1}a_i^{-1}x \in H$, so $xH = a_ib_jH$.

4. Let H be the subgroupp generated by σ and τ. Since $\sigma^{-1}(i\ i+1)\sigma = (i-1\ i)$, all cycles of length 2 of the form $(i\ i+1)$, $1 \le i \le n-1$, are in H. We prove by induction on $j-i$, that every cycle of the form $(i\ j), i < j$, is in H. If $j-i = 1$, this is true by the above argument. Assume the statement is true if $j-i \le k-1$, then if $j-i = k$, we have

$$(i\ j) = (i\ j-1)(j-1\ j)(i\ j-1),$$

so $(i\ j)$ is also in H.

We have shown that every 2-cycle is in H, and since every cycles can be written as a product of 2-cycles, it follows that every cycle is in H.

5. Every subgroup of an abelian group is normal. This exercise shows that the converse is not true.

(i) Let I denote the identity matrix. We have $A^2 = -I$, and so $A^3 = -A$, and $A^4 = I$. Similarly, $B^2 = -I$, so $B^3 = -B$, and $B^4 = I$. Therefore, A and B are both of order 4. Also,

$$AB = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -BA.$$

So the group generated by A and B has 8 elements $\{I, -I, A, A^3, B, B^3, AB, -AB\}$. Clearly this is closed under inverse ($A^{-1} = A^3, B^{-1} = B^3, (AB)^{-1} = (-AB)$, and $(-I)^{-1} = -I$), and since $AB = -BA$, it is closed under multiplication. The group G is not abelian. Since $AB \neq BA$.

(ii) Every non-trivial subgroup of G has order 2 or 4. Since every subgroup of order 4 has index 2, it is normal by Exercise 3. So we need to show all the subgroups of order 2 are normal. Subgroups of order 2 are generated by elements of order 2, and G has only 2 elements of order 2: A^2 and B^2. Let $G =< A^2 >= \{I, -I\}$. For every $C \in SL(2, \mathbf{C})$, $C(-I)C^{-1} = -CC^{-1} = -I \in H$. So H is normal. Similarly the group generated by B^2 is normal.

6. By the second isomorphism theorem, we have

$$HN/N \simeq H/H \cap N.$$

Denote by n the order of the above isomorphic groups. Then $n||H|$ and since NH/N is a subgroup of G/N, n divides $|G/N| = [G : N]$. Since $|H|$ and $[G : N]$ are relatively prime, $n = 1$, so $H \cap N = H$ and so $H \leq N$.

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(i), (ii) Note that $[a, b]^{-1} = [b, a]$ so G' is the subset of G consisting of finite product of commutators

$$G' = \{[a_1, b_1][a_2, b_2] \dots [a_n, b_n] | a_i, b_i \in G, n \geq 1\}.$$

So to show that G' is normal, it is enough to show for every $c \in G$, $c^{-1}[a, b]c \in G'$. We can write

$$c^{-1}[a, b]c = c^{-1}a^{-1}b^{-1}abc = [c^{-1}ac, c^{-1}bc] \in G'.$$

so

$$c^{-1}[a_1, b_1] \dots [a_n, b_n]c = [c^{-1}a_1c, c^{-1}b_1c] \dots [c^{-1}a_nc, c^{-1}b_nc] \in G'.$$

Now we prove a more general statement: For any normal subgroup N of G, G/N is abelian if and only if G' is contained in N. We have G/N is abelian $\Leftrightarrow$ $aNbN = bNaN$ for all $a, b \in G \Leftrightarrow abN = baN$ for all $a, b \in G \Leftrightarrow a^{-1}b^{-1}ab \in N$ for all $ab \in G \Leftrightarrow G'$ is a subset of N.

(iii) Of course if $G^{(m)} = \{e\}$,

$$\{e\} = G^{(m)} \leq G^{(m-1)} \leq \cdots \leq G^{(0)} = G$$

is an abelian tower.

Conversely if G is solvable and

$$\{e\} = G_m \leq G_{m-1} \leq \cdots \leq G_0 = G$$

is an abelian tower, then since G/G_1 is abelian, by part (ii), $G' \subset G_1$, and since G_1/G_2 is abelian, $G_1' \subset G_2$, so $G'' \subset G_2$. Continuing this, we see that $G^{(i)} \subset G_i$ for every i, and therefore, $G^{(m)} \subset G_m = \{e\}$, so G is solvable.