Algebra I, Fall 2016

Solutions to Problem Set 10

1. Let $\alpha = \frac{p(t)}{q(t)}$, such that $p(t), q(t) \in K[t]$ and $q(t) \neq 0$. We can assume $p(t)$ and $q(t)$ are relatively prime in $K[t]$. Suppose $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is the minimal polynomial of α over K. Then

$$p^n + a_{n-1}p^{n-1}q + \cdots + a_1 pa^{n-1} + a_0 q^n = 0.$$

This implies that q divides p^n which is only possible if q is a constant (since the gcd of p and q is a unit in $K[t]$.) Also p divides $a_0 q^n$ which is again possible only if $a_0 = 0$ or p is a constant. If $a_0 = 0$, then $f(x) = x(x^{n-1} + \cdots + a_1)$ which contradicts the fact that $f(x)$ is a minimal polynomial and is therefore irreducible. So p and q are both constants, so $\alpha \in K$.

2.

(i) $x^2 - 2$ is irreducible over $\mathbf{Q}$, and the roots are $\pm\sqrt{2}$, so the splitting field is $\mathbf{Q}(\sqrt{2})$ which is a degree 2 extension of $\mathbf{Q}$.

(ii) $x^3 - 2$ is irreducible over $\mathbf{Q}$. If ω is a complex third root of 1 (so a root of $x^2 + x + 1$), then the roots of $x^3 - 2$ are $\sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$. So if E is the splitting field of $x^3 - 2$, then $\omega \in E$, and $E = \mathbf{Q}(\sqrt[3]{2}, \omega)$. If $K = \mathbf{Q}(\sqrt[3]{2})$, then $[K : \mathbf{Q}] = 3$, and $x^2 + x + 1$ is irreducible over K since $\omega \notin \mathbf{Q}(\sqrt[3]{2})$, so $[E : \mathbf{Q}] = 6$.

(iii) $x^2 + x + 1$ is irreducible over $\mathbf{Q}$. The roots are ω and ω^2, so the splitting field is $\mathbf{Q}(\omega)$ which is of degree 2 over $\mathbf{Q}$.

3. If $\overline{K}$ is the algebraic closure of K, then every $\alpha \in \overline{K}$ is the root of a polynomial in K, but there are countably many such polynomials, and each such polynomial has finitely many roots, so the roots of all polynomials in $K[x]$ form a countable set. Note that a finite field F cannot be algebraically closed since if $F = \{a_1, \ldots, a_n\}$, then the

polynomial $f(x) = (x - a_1)(x - a_2) \dots (x - a_n) + 1$ has no root in F. So $\overline{K}$ is not finite.

4. Let $a \in F$, and consider the splitting field E of $f(x) = x^p - a$ over F. Then E is of characteristic p. Let $b \in E$ be a root of this polynomial. then $b^p = a$, so

$$f(x) = x^p - a = x^p - b^p = (x - b)^p.$$

So the only root of $f(x)$ in E is b. Now if

$$f(x) = f_1(x) \dots f_k(x)$$

is the factorization of f into irreducible polynomials in $F[x]$, then since f is separable, each f_i should be separable, so f_i cannot have repeated roots, but the only root of each of the f_i is b, so each b_i has to be of degree 1 in $F[x]$. Therefore the root b of these degree 1 polynomials should be also in F, hence $b \in F$.