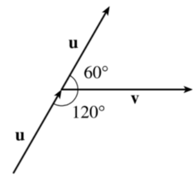
14. Using Theorem 9, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = (4)(5)\sin 45^\circ = 20\cdot\frac{\sqrt{2}}{2} = 10\sqrt{2}$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed out of the page.

15. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 60°. Using Theorem 9, we have

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = (12)(16)\sin 60^\circ = 192\cdot\frac{\sqrt{3}}{2} = 96\sqrt{3}.$$

By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.



17. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}\mathbf{k} = (-1-6)\,\mathbf{i} - (2-12)\,\mathbf{j} + [4-(-4)]\,\mathbf{k} = -7\,\mathbf{i} + 10\,\mathbf{j} + 8\,\mathbf{k}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{k} = [6-(-1)]\,\mathbf{i} - (12-2)\,\mathbf{j} + (-4-4)\,\mathbf{k} = 7\,\mathbf{i} - 10\,\mathbf{j} - 8\,\mathbf{k}$$

Notice $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Property 1 of the cross product.

18. $$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k} = 4\,\mathbf{i} - 6\,\mathbf{j} + 2\,\mathbf{k} \text{ so}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 4 & -6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -6 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 4 & -6 \end{vmatrix}\mathbf{k} = 6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}.$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k} \text{ so}$$

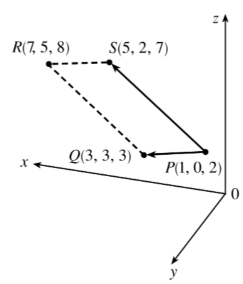
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}\mathbf{k} = 8\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}.$$

Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

28. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{PQ} = \langle 2,3,1\rangle$ and $\overrightarrow{PS} = \langle 4,2,5\rangle$. Thus the area of parallelogram $PQRS$ is

$$\left|\overrightarrow{PQ}\times\overrightarrow{PS}\right| = \left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{matrix}\right\| = |(15-2)\,\mathbf{i} - (10-4)\,\mathbf{j} + (4-12)\,\mathbf{k}|$$

$$= |13\,\mathbf{i} - 6\,\mathbf{j} - 8\,\mathbf{k}| = \sqrt{169+36+64} = \sqrt{269} \approx 16.40$$



29. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -3,1,2\rangle$ and $\overrightarrow{PR} = \langle 3,2,4\rangle$, so

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (1)(4)-(2)(2),\,(2)(3)-(-3)(4),\,(-3)(2)-(1)(3)\rangle = \langle 0,18,-9\rangle$$

Therefore, $\langle 0,18,-9\rangle$ (or any nonzero scalar multiple thereof, such as $\langle 0,2,-1\rangle$) is orthogonal to the plane through P, Q, and R.

(b) Note that the area of the triangle determined by P, Q, and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is

$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 0,18,-9\rangle| = \sqrt{0+324+81} = \sqrt{405} = 9\sqrt{5}$, so the area of the triangle is $\frac{1}{2}\cdot 9\sqrt{5} = \frac{9}{2}\sqrt{5}$.

14. Using Theorem 9, we have $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta = (4)(5)\sin 45^\circ = 20 \cdot \dfrac{\sqrt{2}}{2} = 10\sqrt{2}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

34. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 + 1 + 0 = 1.$

So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is 1 cubic unit.

38. $\mathbf{u} = \overrightarrow{AB} = \langle 2, -4, 4\rangle$, $\mathbf{v} = \overrightarrow{AC} = \langle 4, -1, -2\rangle$ and $\mathbf{w} = \overrightarrow{AD} = \langle 2, 3, -6\rangle$.

$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2 \begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix} - (-4) \begin{vmatrix} 4 & -2 \\ 2 & -6 \end{vmatrix} + 4 \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 24 - 80 + 56 = 0$, so the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 0, which says these vectors lie in the same plane. Therefore, their initial and terminal points A, B, C and D also lie in the same plane.

43. From Theorem 9 we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and from Theorem 12.3.3 we have $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta \quad \Rightarrow \quad |\mathbf{a}|\,|\mathbf{b}| = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\cos\theta}$. Substituting the second equation into the first gives $|\mathbf{a} \times \mathbf{b}| = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\cos\theta} \sin\theta$, so $\dfrac{|\mathbf{a} \times \mathbf{b}|}{\mathbf{a} \cdot \mathbf{b}} = \tan\theta$. Here $|\mathbf{a} \times \mathbf{b}| = |\langle 1, 2, 2\rangle| = \sqrt{1 + 4 + 4} = 3$, so $\tan\theta = \dfrac{|\mathbf{a} \times \mathbf{b}|}{\mathbf{a} \cdot \mathbf{b}} = \dfrac{3}{\sqrt{3}} = \sqrt{3} \quad \Rightarrow \quad \theta = 60^\circ$.

44. (a) Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. Then

$$\langle 1, 2, 1\rangle \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} = (2v_3 - v_2)\,\mathbf{i} - (v_3 - v_1)\,\mathbf{j} + (v_2 - 2v_1)\,\mathbf{k}.$$

If $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$ then $\langle 2v_3 - v_2, v_1 - v_3, v_2 - 2v_1\rangle = \langle 3, 1, -5\rangle \quad \Leftrightarrow \quad 2v_3 - v_2 = 3$ **(1)**, $v_1 - v_3 = 1$ **(2)**, and $v_2 - 2v_1 = -5$ **(3)**. From **(3)** we have $v_2 = 2v_1 - 5$ and from **(2)** we have $v_3 = v_1 - 1$; substitution into **(1)** gives $2(v_1 - 1) - (2v_1 - 5) = 3 \quad \Rightarrow \quad 3 = 3$, so this is a dependent system. If we let $v_1 = a$ then $v_2 = 2a - 5$ and $v_3 = a - 1$, so $\mathbf{v}$ is any vector of the form $\langle a, 2a - 5, a - 1\rangle$.

(b) If $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, 5\rangle$ then $2v_3 - v_2 = 3$ **(1)**, $v_1 - v_3 = 1$ **(2)**, and $v_2 - 2v_1 = 5$ **(3)**. From **(3)** we have $v_2 = 2v_1 + 5$ and from **(2)** we have $v_3 = v_1 - 1$; substitution into **(1)** gives $2(v_1 - 1) - (2v_1 + 5) = 3 \quad \Rightarrow \quad -7 = 3$, so this is an inconsistent system and has no solution.

Alternatively, if we use matrices to solve the system we could show that the determinant is 0 (and hence the system has no solution).