10. \( f(x, y, z) = y^2 e^{xyz} \)
   (a) \( \nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)) = (y^2 e^{xyz}(yz), y^2 \cdot e^{xyz}(xz) + e^{xyz} \cdot 2y, y^2 e^{xyz}(xy)) \)
   = \( (y^2 z e^{xyz}, (xy^2 z + 2y) e^{xyz}, x y^3 e^{xyz}) \)
   (b) \( \nabla f(0, 1, -1) = (-1, 2, 0) \)
   (c) \( D_u f(0, 1, -1) = \nabla f(0, 1, -1) \cdot u = (-1, 2, 0) \cdot (\frac{4}{15}, \frac{1}{15}, \frac{16}{15}) = -\frac{4}{15} + \frac{2}{15} + 0 = \frac{6}{15} \)

15. \( f(x, y, z) = x^2 y + y^2 z \Rightarrow \nabla f(x, y, z) = (2xy, x^2 + 2yz, y^2) \), \( \nabla f(1, 2, 3) = (4, 13, 4) \), and a unit vector in the direction of \( \mathbf{v} \) is \( u = \frac{1}{\sqrt{4 + 1 + 4}} (2, -1, 2) = \frac{1}{3} (2, -1, 2) \), so
   \( D_u f(1, 2, 3) = \nabla f(1, 2, 3) \cdot u = (4, 13, 4) \cdot \frac{1}{3} (2, -1, 2) = \frac{1}{3} (8 - 13 + 8) = \frac{1}{3} = 1 \).

21. \( f(x, y) = 4y \sqrt{x} \Rightarrow \nabla f(x, y) = (4y \cdot \frac{1}{2} x^{-\frac{1}{2}}, 4 \sqrt{x}) = (2y \sqrt{x}, 4 \sqrt{x}) \).
   \( \nabla f(4, 1) = (1, 8) \) is the direction of maximum rate of change, and the maximum rate is \( |\nabla f(4, 1)| = \sqrt{1 + 64} = \sqrt{65} \).

22. \( f(s, t) = te^t \Rightarrow \nabla f(s, t) = (te^t(t), te^t(s) + e^t(1)) = (t e^t, (st + 1)e^t) \).
   \( \nabla f(0, 2) = (4, 1) \) is the direction of maximum rate of change, and the maximum rate is \( |\nabla f(0, 2)| = \sqrt{16 + 1} = \sqrt{17} \).

29. The direction of fastest change is \( \nabla f(x, y) = (2x - 2) i + (2y - 4) j \), so we need to find all points \( (x, y) \) where \( \nabla f(x, y) \) is parallel to \( i + j \) \( \Leftrightarrow \) \( (2x - 2) i + (2y - 4) j = k (i + j) \Leftrightarrow k = 2x - 2 \) and \( k = 2y - 4 \Rightarrow 2x - 2 = 2y - 4 \Rightarrow y = x + 1 \), so the direction of fastest change is \( i + j \) at all points on the line \( y = x + 1 \).

42. Let \( F(x, y, z) = y^2 + z^2 - x \). Then \( x = y^2 + z^2 + 1 \Leftrightarrow y^2 + z^2 - x = -1 \) is a level surface of \( F \).
   \( F_x(x, y, z) = -1 \Rightarrow F_x(3, 1, -1) = -1, \) \( F_y(x, y, z) = 2y \Rightarrow F_y(3, 1, -1) = 2, \) and \( F_z(x, y, z) = 2z \Rightarrow F_z(3, 1, -1) = -2 \).
   (a) By Equation 19, an equation of the tangent plane at \( (3, 1, -1) \) is \( (-1)(x - 3) + 2(y - 1) + (-2)[z - (-1)] = 0 \) or
   \( -x + 2y - 2z = 1 \) or \( x - 2y + 2z = -1 \).
   (b) By Equation 20, the normal line has symmetric equations \( \frac{x - 3}{-1} = \frac{y - 1}{2} = \frac{z + 1}{-2} \) and equivalent equations \( x = 3 - t, y = 1 + 2t, z = -1 - 2t \).

43. Let \( F(x, y, z) = xy^2 z^3 \). Then \( xy^2 z^3 = 8 \) is a level surface of \( F \) and \( \nabla F(x, y, z) = (y^2 z^3, 2xyz^3, 3xyz^2 z^2) \).
   (a) \( \nabla F(2, 2, 1) = (4, 8, 24) \) is a normal vector for the tangent plane at \( (2, 2, 1) \), so an equation of the tangent plane is
   \( 4(x - 2) + 8(y - 2) + 24(z - 1) = 0 \) or \( 4x + 8y + 24z = 48 \) or equivalently \( x + 2y + 6z = 12 \).
   (b) The normal line has direction \( \nabla F(2, 2, 1) = (4, 8, 24) \) or equivalently \( (1, 2, 6) \), so parametric equations are \( x = 2 + t, y = 2 + 2t, z = 1 + 6t \), and symmetric equations are \( x - 2 = \frac{y - 2}{2} = \frac{z - 1}{6} \).

44. Let \( F(x, y, z) = xy + yz + zx \). Then \( xy + yz + zx = 5 \) is a level surface of \( F \) and \( \nabla F(x, y, z) = (y + z, x + z, x + y) \).
   (a) \( \nabla F(1, 2, 1) = (3, 2, 3) \) is a normal vector for the tangent plane at \( (1, 2, 1) \), so an equation of the tangent plane
   is \( 3(x - 1) + 2(y - 2) + 3(z - 1) = 0 \) or \( 3x + 2y + 3z = 10 \).
   (b) The normal line has direction \( (3, 2, 3) \), so parametric equations are \( x = 1 + 3t, y = 2 + 2t, z = 1 + 3t \), and symmetric equations are \( \frac{x - 1}{3} = \frac{y - 2}{2} = \frac{z - 1}{3} \).
54. Let \( F(x, y, z) = x^2 + y^2 + 2z^2 \); then the ellipsoid \( x^2 + y^2 + 2z^2 = 1 \) is a level surface of \( F \). \( \nabla F(x, y, z) = (2x, 2y, 4z) \) is a normal vector to the surface at \((x, y, z)\) and so it is a normal vector for the tangent plane there. The tangent plane is parallel to the plane \( x + 2y + z = 1 \) when the normal vectors of the planes are parallel, so we need a point \((x_0, y_0, z_0)\) on the ellipsoid where \((2x_0, 2y_0, 4z_0) = k(1, 2, 1)\) for some \( k \neq 0 \). Comparing components we have \( 2x_0 = k \Rightarrow x_0 = k/2, \)
\[
2y_0 = 2k \quad \Rightarrow \quad y_0 = k, \quad 4z_0 = k \quad \Rightarrow \quad z_0 = k/4.
\]
Thus \((x_0, y_0, z_0) = (k/2, k, k/4)\) lies on the ellipsoid, so
\[
(k/2)^2 + k^2 + 2(k/4)^2 = 1 \quad \Rightarrow \quad \frac{11}{16}k^2 = 1 \quad \Rightarrow \quad k^2 = \frac{8}{11} \quad \Rightarrow \quad k = \pm 2\sqrt{\frac{2}{11}}.
\]
Thus the tangent planes at the points \((\sqrt{\frac{2}{11}}, 2\sqrt{\frac{2}{11}}, \frac{1}{2}\sqrt{\frac{2}{11}})\) and \((-\sqrt{\frac{2}{11}}, -2\sqrt{\frac{2}{11}}, -\frac{1}{2}\sqrt{\frac{2}{11}})\) are parallel to the given plane.

57. Let \((x_0, y_0, z_0)\) be a point on the cone \([other than (0, 0, 0)]\). The cone is a level surface of \( F(x, y, z) = x^2 + y^2 - z^2 \) and \( \nabla F(x, y, z) = (2x, 2y, -2z) \), so \( \nabla F(x_0, y_0, z_0) = (2x_0, 2y_0, -2z_0) \) is a normal vector to the cone at this point and an equation of the tangent plane there is \( 2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0 \) or
\[
x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2 = \text{constant}.
\]
But \( x_0^2 + y_0^2 = z_0^2 \) so the tangent plane is given by \( x_0x + y_0y - z_0z = 0 \), a plane which always contains the origin.

58. Let \((x_0, y_0, z_0)\) be a point on the sphere. Then the normal line is given by \( \frac{x - x_0}{2x_0} = \frac{y - y_0}{2y_0} = \frac{z - z_0}{2z_0} \). For the center \((0, 0, 0)\) to be on the line, we need \( -\frac{x_0}{2x_0} = -\frac{y_0}{2y_0} = -\frac{z_0}{2z_0} \) or equivalently \( 1 = 1 = 1 \), which is true.

60. The ellipsoid is a level surface of \( F(x, y, z) = 4x^2 + y^2 + 4z^2 \) and \( \nabla F(x, y, z) = (8x, 2y, 8z) \), so \( \nabla F(1, 2, 1) = (8, 4, 8) \) or equivalently \((2, 1, 2)\) is a normal vector to the surface. Thus the normal line to the ellipsoid at \((1, 2, 1)\) is given by \( x = 1 + 2t, \ y = 2 + t, \ z = 1 + 2t \). Substitution into the equation of the sphere gives
\[
(1 + 2t)^2 + (2 + t)^2 + (1 + 2t)^2 = 102 \quad \Leftrightarrow \quad 6 + 12t + 9t^2 = 102 \quad \Leftrightarrow \quad 9t^2 + 12t - 96 = 0 \quad \Leftrightarrow \quad 3(t + 4)(3t - 8) = 0.
\]
Thus the line intersects the sphere when \( t = -4 \), corresponding to the point \((-7, -2, -7)\), and when \( t = \frac{8}{3} \), corresponding to the point \((\frac{10}{3}, \frac{14}{3}, \frac{10}{3})\).