7. \( f(x, y) = (x - y)(1 - xy) = x - y - x^2 y + xy^2 \Rightarrow f_x = 1 - 2xy + y^2, f_y = -1 - x^2 + 2xy, f_{xx} = -2y, f_{xy} = -2x + 2y, f_{yy} = 2x. \) Then \( f_x = 0 \) implies \( 1 - 2xy + y^2 = 0 \) and \( f_y = 0 \) implies \( -1 - x^2 + 2xy = 0 \). Adding the two equations gives \( 1 + y^2 - x^2 = 0 \Rightarrow y^2 = x^2 \Rightarrow y = \pm x, \) but if \( y = -x \) then \( f_x = 0 \) implies \( 1 + 2x^2 + x^2 = 0 \Rightarrow 3x^2 = -1 \) which has no real solution. If \( y = x \) then substitution into \( f_x = 0 \) gives \( 1 - 2x^2 + x^2 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \), so the critical points are \((1, 1)\) and \((-1, -1)\). Now

\[
D(1, 1) = (-2)(2) - 0^2 = -4 < 0 \text{ and } D(-1, -1) = (2)(-2) - 0^2 = -4 < 0, \text{ so } (1, 1) \text{ and } (-1, -1) \text{ are saddle points.}
\]

8. \( f(x, y) = y(e^y - 1) \Rightarrow f_x = ye^y, f_y = e^y - 1, f_{xx} = ye^y, f_{xy} = e^y, f_{yy} = 0. \) Because \( e^y \) is never zero, \( f_x = 0 \) only when \( y = 0 \), and \( f_y = 0 \) when \( e^y = 1 \Rightarrow x = 0 \), so the only critical point is \((0, 0)\).

\[
D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = (ye^y)(0) - (e^y)^2 = -e^{2y}, \text{ and since } D(0, 0) = -1 < 0, \text{ (0, 0) is a saddle point.}
\]

21. \( f(x, y) = x^2 + 4y^2 - 4xy + 2 \Rightarrow f_x = 2x - 4y, f_y = 8y - 4x, f_{xx} = 2, f_{xy} = -4, f_{yy} = 8. \) Then \( f_x = 0 \) and \( f_y = 0 \) each implies \( y = \frac{1}{2}x \), so all points of the form \((x_0, \frac{1}{2}x_0)\) are critical points and for each of these we have

\[
D(x_0, \frac{1}{2}x_0) = (2)(8) - (-4)^2 = 0. \text{ The Second Derivatives Test gives no information, but } f(x, y) = x^2 + 4y^2 - 4xy + 2 = (x - 2y)^2 + 2 \geq 2 \text{ with equality if and only if } y = \frac{1}{2}x. \text{ Thus } f(x_0, \frac{1}{2}x_0) = 2 \text{ are all local (and absolute) minima.}
\]

32. Since \( f \) is a polynomial it is continuous on \( D \), so an absolute maximum and minimum exist. \( f_x = 1 - y, f_y = 1 - x, \) and setting \( f_x = f_y = 0 \) gives \((1, 1)\) as the only critical point (which is inside \( D \)), where \( f(1, 1) = 1 \). Along \( L_1: y = 0 \) and \( f(x, 0) = x \) for \( 0 \leq x \leq 4 \), an increasing function in \( x \), so the maximum value is \( f(4, 0) = 4 \) and the minimum value is \( f(0, 0) = 0. \) Along \( L_2: x = 2 - \frac{1}{2} \) and \( f(x, 2 - \frac{1}{2}x) = \frac{1}{2}x^2 - \frac{3}{2}x + 2 = \frac{1}{2}(x - \frac{3}{2})^2 + \frac{7}{8} \) for \( 0 \leq x \leq 4 \), a quadratic function which has a minimum at \( x = \frac{3}{2} \), where \( f\left(\frac{3}{2}, \frac{5}{2}\right) = \frac{7}{8} \), and a maximum at \( x = 4, \) where \( f(4, 0) = 4. \)

Along \( L_3: x = 0 \) and \( f(0, y) = y \) for \( 0 \leq y \leq 2, \) an increasing function in \( y, \) so the maximum value is \( f(0, 2) = 2 \) and the minimum value is \( f(0, 0) = 0. \) Thus the absolute maximum of \( f \) on \( D \) is \( f(4, 0) = 4 \) and the absolute minimum is \( f(0, 0) = 0. \)

33. \( f_x(x, y) = 2x + 2xy, f_y(x, y) = 2y + x^2, \) and setting \( f_x = f_y = 0 \) gives \((0, 0)\) as the only critical point in \( D, \) with \( f(0, 0) = 4. \)

On \( L_1: y = -1, f(x, -1) = 5, \) a constant.

On \( L_2: x = 1, f(1, y) = y^2 + y + 5, \) a quadratic in \( y \) which attains its maximum at \((1, 1), \) \( f(1, 1) = 7 \) and its minimum at \((1, -\frac{1}{2}), \) \( f\left(1, -\frac{1}{2}\right) = \frac{19}{4} \).

On \( L_3: f(1, 1) = 2x^2 + 5 \) which attains its maximum at \((-1, 1)\) and \((1, 1)\) with \( f(\pm 1, 1) = 7 \) and its minimum at \((0, 1), \) \( f(0, 1) = 5. \)

On \( L_4: f(-1, 1) = y^2 + y + 5 \) with maximum at \((-1, 1), \) \( f(-1, 1) = 7 \) and minimum at \((-1, -\frac{1}{2}), \) \( f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}. \)

Thus the absolute maximum is attained at both \((\pm 1, 1)\) with \( f(\pm 1, 1) = 7 \) and the absolute minimum on \( D \) is attained at \((0, 0)\) with \( f(0, 0) = 4. \)
36. \( f_x = y^2 \) and \( f_y = 2xy \), and since \( f_z = 0 \) \( \iff \) \( y = 0 \), there are no critical points in the interior of \( D \). Along \( L_1 \): \( y = 0 \) and \( f(x, 0) = 0 \).

Along \( L_2 \): \( x = 0 \) and \( f(0, y) = 0 \). Along \( L_3 \): \( y = \sqrt{3} - x^2 \), so let
\[
g(x) = f(x, \sqrt{3} - x^2) = 3x - x^3 \quad \text{for} \quad 0 \leq x \leq \sqrt{3}.
\]
Then
\[
g'(x) = 3 - 3x^2 = 0 \quad \iff \quad x = 1.
\]
The maximum value is \( f(1, \sqrt{2}) = 2 \) and the minimum occurs both at \( x = 0 \) and \( x = \sqrt{3} \) where
\[
f(0, \sqrt{3}) = f(\sqrt{3}, 0) = 0.
\]
Thus the absolute maximum of \( f \) on \( D \) is \( f(1, \sqrt{2}) = 2 \), and the absolute minimum is 0 which occurs at all points along \( L_1 \) and \( L_2 \).

44. The distance from the origin to a point \((x, y, z)\) on the surface is \( d = \sqrt{x^2 + y^2 + z^2} \) where \( y^2 = 9 + xz \), so we minimize \( d^2 = x^2 + 9 + xz + z^2 = f(x, z) \). Then \( f_x = 2x + z, f_z = x + 2z \), and \( f_x = 0, f_z = 0 \) \( \iff \) \( x = 0, z = 0 \), so the only critical point is \((0, 0)\). \( D(0, 0) = (2/2) - 1 = 3 > 0 \) with \( f_{xx}(0, 0) = 2 > 0 \), so this is a minimum. Thus \( y^2 = 9 + 0 \) \( \iff \) \( y = \pm 3 \) and the points on the surface closest to the origin are \((0, \pm 3, 0)\).

45. Let \( x, y, z \) be the positive numbers. Then \( x + y + z = 100 \) \( \implies \) \( z = 100 - x - y \), and we want to maximize \( xyz = xy(100 - x - y) = 100xy - x^2y - xy^2 = f(x, y) \) for \( 0 < x, y, z < 100 \). \( f_x = 100y - 2xy - y^2 \),
\[
f_y = 100x - x^2 - 2xy, \quad f_{xx} = -2y, \quad f_{yy} = -2x, \quad f_{xy} = 100 - 2x - 2y. \]
Then \( f_x = 0 \) implies \( y(100 - 2x - y) = 0 \) \( \iff \) \( y = 100 - 2x \) (since \( y > 0 \)). Substituting into \( f_y = 0 \) gives \( x[100 - x - 2(100 - 2x)] = 0 \) \( \iff \) \( 3x - 100 = 0 \) (since \( x > 0 \)) \( \iff \) \( x = \frac{100}{3} \). Then \( y = 100 - 2 \left( \frac{100}{3} \right) = \frac{100}{3} \), and the only critical point is \((\frac{100}{3}, \frac{100}{3})\).
\[
D\left(\frac{100}{3}, \frac{100}{3}\right) = \left( -\frac{200}{9} \right) - \left( -\frac{100}{3} \right)^2 = \frac{10000}{9} > 0 \quad \text{and} \quad f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0. \quad \text{Thus} \quad f\left(\frac{100}{3}, \frac{100}{3}\right)
\]
is a local maximum. It is also the absolute maximum (compare to the values of \( f \) as \( x, y, \text{or} z \to 0 \text{ or } 100 \)), so the numbers are \( x = y = z = \frac{100}{3} \).

46. Let \( x, y, z \), be the positive numbers. Then \( x + y + z = 12 \) and we want to minimize
\[
x^2 + y^2 + z^2 = x^2 + y^2 + (12 - x - y)^2 = f(x, y) \quad \text{for} \quad 0 < x, y < 12. \quad f_x = 2x + 2(12 - x - y)(-1) = 4x + 2y - 24,
\]
\[
f_y = 2y + 2(12 - x - y)(-1) = 2x + 4y - 24, \quad f_{xx} = 4, \quad f_{xy} = 2, \quad f_{yy} = 4. \quad \text{Then} \quad f_x = 0 \implies 4x + 2y = 24 \quad \text{or} \quad y = 12 - 2x \quad \text{and substituting into} \quad f_y = 0 \quad \text{gives} \quad 2x + 4(12 - 2x) = 24 \quad \iff \quad 6x = 24 \quad \iff \quad x = 4 \quad \text{and then} \quad y = 4, \quad \text{so the only critical point is} \quad (4, 4). \quad D(4, 4) = 16 - 4 > 0 \quad \text{and} \quad f_{xx}(4, 4) = 4 > 0, \quad \text{so} \quad f(4, 4) \quad \text{is a local minimum.} \quad f(4, 4) \quad \text{is also the absolute minimum} \quad \text{[compare to the values of} \quad f \quad \text{as} \quad x, y, \text{or} \quad z \to 0 \text{ or } 12 \text{]} \quad \text{so the numbers are} \quad x = y = z = 4. \]

48. Let \( x, y, z \) be the dimensions of the box. We wish to minimize surface area \( = 2xy + 2xz + 2yz \), but we have
\[
\text{volume} = xyz = 1000 \quad \iff \quad z = \frac{1000}{xy} \quad \text{so we minimize}
\]
\[
f(x, y) = 2xy + 2x \left( \frac{1000}{xy} \right) + 2y \left( \frac{1000}{xy} \right) = 2xy + \frac{2000}{y} + \frac{2000}{x}. \quad \text{Then} \quad f_x = 2y - \frac{2000}{x^2} \quad \text{and} \quad f_y = 2x - \frac{2000}{y^2}. \quad \text{Setting}
\]
\[
f_x = 0 \implies y = \frac{1000}{x^2} \quad \text{and substituting into} \quad f_y = 0 \quad \text{gives} \quad x - \frac{x^4}{1000} = 0 \quad \iff \quad x^3 = 1000 \quad \text{[since} \quad x \neq 0] \quad \iff \quad x = 10.
\]
The surface area has a minimum but no maximum and it must occur at a critical point, so the minimal surface area occurs for a box with dimensions \( x = 10 \text{ cm}, \quad y = 1000/10^2 = 10 \text{ cm}, \quad z = 1000/10^2 = 10 \text{ cm}. \)