7. $\partial (ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x \cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function $f$ such that $\nabla f = \mathbf{F}$. Then $f_x(x, y) = ye^x + \sin y$ implies $f(x, y) = ye^x + x \sin y + g(y)$ and $f_y(x, y) = e^x + x \cos y + g'(y)$. But $f_y(x, y) = e^x + x \cos y$ so $g(y) = K$ and $f(x, y) = ye^x + x \sin y + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\frac{\partial}{\partial y} (2xy) = 2x = \frac{\partial}{\partial x} (x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$ depends only on the endpoints of $C$. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F} \cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function $f$, so that $\nabla f = \mathbf{F}$. We know $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Integrating $f_x(x, y)$ with respect to $x$, we have $f(x, y) = x^2y + g(y)$. Differentiating both sides with respect to $y$ gives $f_y(x, y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \Rightarrow g'(y) = 0 \Rightarrow g(y) = K$, a constant.

Thus $f(x, y) = x^2y + K$, and we can take $K = 0$. All three curves start at $(1, 2)$ and end at $(3, 2)$, so by Theorem 2, $\int_C \mathbf{F} \cdot d\mathbf{r} = f(3, 2) - f(1, 2) = 18 - 2 = 16$ for each curve.

15. (a) $f_x(x, y, z) = yz$ implies $f(x, y, z) = xyz + g(y, z)$ and so $f_y(x, y, z) = xz + g_y(y, z)$. But $f_y(x, y, z) = xz$ so $g_y(y, z) = 0 \Rightarrow g(y, z) = h(z)$. Thus $f(x, y, z) = xyz + h(z)$ and $f_x(x, y, z) = xy + h'(z)$. But $f_x(x, y, z) = xy + 2z$, so $h'(z) = 2z \Rightarrow h(z) = z^2 + K$. Hence $f(x, y, z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 6, 3) - f(1, 0, -2) = 81 - 4 = 77$.

19. The functions $2xe^{-y}$ and $2y - x^2e^{-y}$ have continuous first-order derivatives on $\mathbb{R}^2$ and

$$\frac{\partial}{\partial y} (2xe^{-y}) = -2xe^{-y} = \frac{\partial}{\partial x} (2y - x^2e^{-y}),$$

so $\mathbf{F}(x, y) = 2xe^{-y} \mathbf{i} + (2y - x^2e^{-y}) \mathbf{j}$ is a conservative vector field by

Theorem 6 and hence the line integral is independent of path. Thus a potential function $f$ exists, and $f_x(x, y) = 2xe^{-y}$ implies $f(x, y) = x^2e^{-y} + g(y)$ and $f_y(x, y) = -2xe^{-y} + g'(y)$. But $f_y(x, y) = 2y - x^2e^{-y}$ so $g'(y) = 2y \Rightarrow g(y) = y^2 + K$. We can take $K = 0$, so $f(x, y) = x^2e^{-y} + y^2$. Then

$$\int_C 2xe^{-y} \, dx + (2y - x^2e^{-y}) \, dy = f(2, 1) - f(1, 0) = 4e^{-1} + 1 - 1 = 4/e.$$ 

25. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path $C$, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. But take $C$ to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on $C$ are roughly in the direction of motion along $C$, so the integral around $C$ will be positive. Therefore the field is not conservative.
29. Since \( \mathbf{F} \) is conservative, there exists a function \( f \) such that \( \mathbf{F} = \nabla f \), that is, \( P = f_x, Q = f_y, \) and \( R = f_z \). Since \( P, \) 
\( Q, \) and \( R \) have continuous first order partial derivatives, Clairaut’s Theorem says that \( \partial P / \partial y = f_{xy} = f_{yx} = \partial Q / \partial x, \) 
\( \partial P / \partial z = f_{zx} = f_{xz} = \partial R / \partial x, \) and \( \partial Q / \partial z = f_{yz} = f_{zy} = \partial R / \partial y. \)

30. Here \( \mathbf{F}(x, y, z) = y \mathbf{i} + x \mathbf{j} + xyz \mathbf{k}. \) Then using the notation of Exercise 29, \( \partial P / \partial z = 0 \) while \( \partial R / \partial x = yz. \) Since these aren’t equal, \( \mathbf{F} \) is not conservative. Thus by Theorem 4, the line integral of \( \mathbf{F} \) is not independent of path.

35. (a) \( P = -\frac{y}{x^2 + y^2}, \partial P = \frac{y^2 - x^2}{(x^2 + y^2)^2} \) and \( Q = \frac{x}{x^2 + y^2}, \partial Q = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \) Thus \( \partial P / \partial y = \frac{\partial Q}{\partial x}. \)

(b) \( C_1: x = \cos t, y = \sin t, 0 \leq t \leq \pi, C_2: x = \cos t, y = \sin t, t = 2\pi \) to \( t = \pi. \) Then

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t} dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi dt = -\pi
\]

Since these aren’t equal, the line integral of \( \mathbf{F} \) isn’t independent of path. (Or notice that \( \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi \) where \( C_3 \) is the circle \( x^2 + y^2 = 1, \) and apply the contrapositive of Theorem 3.) This doesn’t contradict Theorem 6, since the domain of \( \mathbf{F}, \) which is \( \mathbb{R}^2 \) except the origin, isn’t simply-connected.