7. $\partial(ye^x + \sin y)/\partial y = e^x + \cos y = \partial(e^x + x\cos y)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Hence $\mathbf{F}$ is conservative so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = ye^x + \sin y$ implies $f(x,y) = ye^x + x\sin y + g(y)$ and $f_y(x,y) = e^x + x\cos y + g'(y)$. But $f_y(x,y) = e^x + x\cos y$ so $g(y) = K$ and $f(x,y) = ye^x + x\sin y + K$ is a potential function for $\mathbf{F}$.

11. (a) $\mathbf{F}$ has continuous first-order partial derivatives and $\dfrac{\partial}{\partial y}(2xy) = 2x = \dfrac{\partial}{\partial x}(x^2)$ on $\mathbb{R}^2$, which is open and simply-connected. Thus, $\mathbf{F}$ is conservative by Theorem 6. Then we know that the line integral of $\mathbf{F}$ is independent of path; in particular, the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C. Since all three curves have the same initial and terminal points, $\int_C \mathbf{F}\cdot d\mathbf{r}$ will have the same value for each curve.

(b) We first find a potential function f, so that $\nabla f = \mathbf{F}$. We know $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Integrating $f_x(x,y)$ with respect to x, we have $f(x,y) = x^2y + g(y)$. Differentiating both sides with respect to y gives $f_y(x,y) = x^2 + g'(y)$, so we must have $x^2 + g'(y) = x^2 \;\Rightarrow\; g'(y) = 0 \;\Rightarrow\; g(y) = K$, a constant.
Thus $f(x,y) = x^2y + K$, and we can take $K = 0$. All three curves start at $(1,2)$ and end at $(3,2)$, so by Theorem 2, $\int_C \mathbf{F}\cdot d\mathbf{r} = f(3,2) - f(1,2) = 18 - 2 = 16$ for each curve.

15. (a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$ so $g_y(y,z) = 0 \;\Rightarrow\; g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$ and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z \;\Rightarrow\; h(z) = z^2 + K$. Hence $f(x,y,z) = xyz + z^2$ (taking $K = 0$).

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

19. The functions $2xe^{-y}$ and $2y - x^2e^{-y}$ have continuous first-order derivatives on $\mathbb{R}^2$ and $\dfrac{\partial}{\partial y}\left(2xe^{-y}\right) = -2xe^{-y} = \dfrac{\partial}{\partial x}\left(2y - x^2e^{-y}\right)$, so $\mathbf{F}(x,y) = 2xe^{-y}\,\mathbf{i} + \left(2y - x^2e^{-y}\right)\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x,y) = 2xe^{-y}$ implies $f(x,y) = x^2e^{-y} + g(y)$ and $f_y(x,y) = -x^2e^{-y} + g'(y)$. But $f_y(x,y) = 2y - x^2e^{-y}$ so $g'(y) = 2y \;\Rightarrow\; g(y) = y^2 + K$. We can take $K = 0$, so $f(x,y) = x^2e^{-y} + y^2$. Then $\int_C 2xe^{-y}\,dx + (2y - x^2e^{-y})\,dy = f(2,1) - f(1,0) = 4e^{-1} + 1 - 1 = 4/e$.

25. We know that if the vector field (call it $\mathbf{F}$) is conservative, then around any closed path C, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$. But take C to be a circle centered at the origin, oriented counterclockwise. All of the field vectors that start on C are roughly in the direction of motion along C, so the integral around C will be positive. Therefore the field is not conservative.

29. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q, and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

30. Here $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 29, $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

35. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi} \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\, dt = \int_0^{\pi} dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^{\pi} dt = -\pi$$

Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.