Algebra I, Fall 2016

Solutions to Problem Set 5

1. (a) IF $ab \in I = f^{-1}(P)$, then $f(ab) \in P$, so $f(a)f(b) \in P$, so $f(a) \in P$ or $f(b) \in P$, so $a \in f^{-1}(P)$ or $b \in f^{-1}f(P)$.

(b) For example, if f is the inclusion of $\mathbf{Z}$ in $\mathbf{Q}$, then since $\mathbf{Q}$ is a field, $\{0\}$ is maximal ideal, but $f^{-1}(\{0\}) = \{0\}$ which is not a maximal ideal in $\mathbf{Z}$.

2. (c) We can get an example by letting $I = J$. For example if $I = J = 2\mathbf{Z}$ in $\mathbf{Z}$, then $IJ = 4\mathbf{Z}$, but $I \cap J = 2\mathbf{Z}$.

4. Let P be a prime ideal and a an element of R which is not in P. Then since R is finite, the elements $\{1, a, a^2, \dots\}$ cannot be all distinct, so there is $i < j$ such that $a^i = a^j$ so $a^i(1 - a^j) = 0$, since P is prime and $0 \in P$, $a^i \in P$ or $(1 - a^j) \in P$. But a^i cannot be in P since a is not in P, so $1 - a^{j-i} \in P$. This implies the ideal generated by P and a contains 1, therefore, $(P, a) = R$ for every a which is not in P. This means P is a maximal ideal.

5. Let P be a maximal ideal among those whose intersection with S is non-empty. Let $ab \in P$. We get a contradiction by assuming a and b are not in S. Since a is not in S, the ideal $I = (P, a) = \{x + ra | \; r \in R, x \in P\}$ contains P but is not equal to P, so $I \cap S \neq \emptyset$, so there is s_1 of the form

$$s_1 = r_1 a + x_1$$

in S. Similarly, if we look at ideal generated by P and b: $J = (P, b)$, we see that there should be an element

$$s_2 = r_2 b + x_2$$

in S. Since S is multiplicative $s_1 s_2 \in S$, so $y = r_1 r_2 ab + r_1 x_2 + r_2 x_1 + x_1 x_2$ is in S, but y is in P since $ab, x_1, x_2 \in P$, contradicting the assumption that $S \cap P = \emptyset$.

6. (a) Primary ideals of $\mathbf{Z}$ are the ideals of the form $p^n\mathbf{Z}$ for a prime number p and a positive integer n: if p^n divides ab, then $p|a$, or $p|b$, so p^n divides a or p^n divides b^n.

(b) If I is primary and $ab \in \sqrt{I}$, then $(ab)^n \in I$ for some $n \geq 1$, so $a^n \in I$ or $b^{nm} \in I$, so $a \in \sqrt{I}$ or $b \in \sqrt{I}$.