1. Let $D$ be the unit disk and $S^1$ the unit circle.

(a) Show that if $g : D \rightarrow \mathbb{C}$ is a continuous function and $g_r : S^1 \rightarrow \mathbb{C}$ is defined by $g_r(z) = g(rz)$, then $g_r(z) \rightarrow g(z)$ uniformly for $z \in S^1$ as $r \rightarrow 1^-$.

(b) If $f : S^1 \rightarrow \mathbb{C}$ is a continuous function, define $\tilde{f} : D \rightarrow \mathbb{C}$ by $\tilde{f}(z) = f(z)$ for $z \in S^1$ and $\tilde{f}(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) P_r(\theta - \phi) \, d\theta$. (So the real and imaginary parts of $\tilde{f}$ are harmonic in $D$.) Define $\tilde{f}_r : S^1 \rightarrow \mathbb{C}$ by $\tilde{f}_r(z) = \tilde{f}(rz)$. Show that for each $r < 1$, there is a sequence $p_n(z, \bar{z})$ of polynomials in $z$ and $\bar{z}$ such that $p_n(z, \bar{z}) \rightarrow \tilde{f}_r(z)$ uniformly for $z \in S^1$. (use Problem 7 of Homework 9.)

(c) **Weierstrass approximation theorem for $S^1$.** If $f : S^1 \rightarrow \mathbb{C}$ is a continuous function, then there is a sequence $p_n(z, \bar{z})$ of polynomials in $z$ and $\bar{z}$ such that $p_n(z, \bar{z}) \rightarrow f(z)$ uniformly for $z \in S^1$.

2. Find a harmonic function on

(a) the unit disk which has boundary values 0 on the lower semicircle and 1 on the upper semicircle.

(b) the first quadrant which has boundary values 0 on $[0, 1]$ and 1 on $[1, \infty]$ and $[0, i\infty]$.

3. Use Fourier coefficients to solve the Dirichlet problem in the unit disk for the function on $[0, 2\pi] : f(\theta) = -1$ if $\pi/2 < \theta < 3\pi/2$ and 1 otherwise.
4. Suppose that \( f \) is an entire function which sends the real line to the real line and the imaginary line to the imaginary line. Prove that \( f \) is an odd function, i.e. \( f(z) = -f(-z) \). (Hint: We showed that if \( f \) sends real line to real line, then \( f(z) = f(\bar{z}) \). Use a similar argument to show that if \( f \) sends the imaginary line to the imaginary line the \( f \) sends points symmetric with respect to the imaginary axis to points symmetric with respect to imaginary axis.)

5. Suppose that \( f(z) \) is holomorphic on \( |z| \leq 1 \) and satisfies \( |f(z)| = 1 \) if \( |z| = 1 \). Show that \( f(z) \) is a rational function.